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The Mitigation, Adaptation and New Technologies Applied General Equilibrium (MANAGE) Model Version 2.0g

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The Mitigation, Adaptation and New Technologies Applied General Equilibrium (MANAGE) Model

Version 2.0g

Dominique van der Mensbrugghe[1]

March 31, 2020

Abstract: This document provides a complete model specification for the **M**itigation, **A**daptation and **N**ew Technologies **A**pplied **G**eneral **E**quilibrium (Manage) Model. Manage is a (recursive) dynamic single country computable general equilibrium (CGE) model designed to focus on energy, emissions and climate change. In addition to the standard features of a single country CGE model, the Manage model includes a detailed energy specification that allows for capital/labor/energy substitution in production, intra-fuel energy substitution across all demand agents, and a multi-output multi-input production structure. Furthermore, Manage is a dynamic model, using by and large the neo-classical growth specification. Labor growth is exogenous. Capital accumulation derives from savings/investment decisions. The model allows for a wide-range of productivity assumptions that include autonomous improvements in energy efficiency that can differ across agents and energy carriers. Finally, the model has a vintage structure for capital that allows for putty/semi-putty assumptions with sluggish mobility of installed capital.

JEL Codes: ¿¿¿¿ INSERT JEL CODES HERE ¡¡¡¡
Keywords: ¿¿¿¿ INSERT KEYWORDS HERE ¡¡¡¡

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Chapter 1

Introduction

The **M**itigation, **A**daptation and **N**ew Technologies **A**pplied **G**eneral **E**quilibrium (MANAGE) Model is a (recursive) dynamic single country computable general equilibrium (CGE) model designed to focus on energy, emissions and climate change. In addition to the standard features of a single country CGE model, the MANAGE model includes a detailed energy specification that allows for capital/labor/energy substitution in production, intra-fuel energy substitution across all demand agents, and a multi-output multi-input production structure. Furthermore, MANAGE is a dynamic model, using by and large the neo-classical growth specification. Labor growth is exogenous. Capital accumulation derives from savings/investment decisions. The model allows for a wide-range of productivity assumptions that include autonomous improvements in energy efficiency that can differ across agents and energy carriers. Finally, the model has a vintage structure for capital that allows for putty/semi-putty assumptions with sluggish mobility of installed capital.

The model is sufficiently flexible that it can be calibrated to a relatively large number of Social Accounting Matrices (SAM), with recent applications available for Morocco and Nigeria. This latest version of the model incorporates recently developed price/volume splits of the energy sectors and CO_2 emissions. The model is implemented in the GAMS software and an aggregation facility is used as a front-end to the model to allow for full aggregation flexibility.

This document is an update of the version 1 and 2a of MANAGE. The key changes include a fuller specification of trade and transport margins, additional indirect taxes, the incorporation of enterprises and a wide variety of monetary transfers—both domestic and international and in the latest version energy volumes and carbon emissions. Version 2.0c includes a new factor of production that is called fixed capital in the sense that it is sector-specific and is not incorporated with the standard capital that is assumed either partially or fully mobile across sectors. The sector-specific capital can be thought of as special equipment that has no economic use in another sector such as a water dam to produce hydro-electricity or solar panels. Version 2.0f has implemented capital/skill substitutability/-complementarity in the production structure. Version 2.0g includes a sector-specific natural resource factor and a price pass-through equation for imports.

Chapter 2

MANAGE in a nutshell

The MANAGE model is a recursive dynamic computable general equilibrium (CGE) model. Each year of a scenario is solved as a static equilibrium, with dynamic equations linking exogenous factors (such as employment growth and capital accumulation) across years with, in addition, update equations for productivity factors. Each static equilibria relies on a relatively standard set of equation specifications.

Production is modeled using a series of nested constant-elasticity-of-substitution (CES) functions designed to capture the substitutions and complements across the different inputs—notably capital and labor, but also with a focus on energy as energy policies are one of the key objectives of the MANAGE model.[1] Energy is assumed to be a near-complement with capital in the short-run, but a substitute in the long-run. Thus rising energy prices tend to lead to rising production costs in the short-run when substitution is low, but a long-run response would lead to energy-saving technologies that dampen the cost-push factor. This feature of the model is embodied in a vintage capital structure that captures the semi-putty/putty relations across inputs with more elastic long-run behavior as compared to the short-run. The model also allows for both multi-input and multi-output production. The former, for example, would allow for electricity supply to be produced by multiple activities—thermal, hydro, solar and other renewable forms of electricity production. The latter allows for a single activity to produce more than one product—for example oil seed crushing produces both vegetable oils and oil cakes (for feed), or corn-based ethanol production can produce both ethanol and distillers dry grains soluble (DDGS) that can be used as a feed substitute.

Labor and capital income is largely allocated to households with pass-through accounts to enterprises. Government revenue is derived from both direct and indirect taxes.

Household demand is modeled using the constant-differences-in-elasticity (CDE) demand function that is the standard utility function used in the GTAP model. The model allows for a different specification of demanded commodities (indexed by k) from supplied commodities (indexed by i). A transition matrix approach is used to convert consumer goods to supplied goods that also relies on a nested CES approach. The transition matrix is largely diagonal in the current version with consumed commodities directly mapped to supplied commodities. Energy demand is bundled into a single commodity and disaggregated by energy type using a CES structure that allows for inter-fuel substitution. Other final demand is handled similarly, though the aggregate expenditure function is a CES function rather than the CDE.

Goods are evaluated at basic prices with tax wedges. The model incorporates trade and transport margins that add an additional wedge between basic prices and end-user prices. The trade and transport margins are differentiated across transport nodes—farm/factory gate to domestic markets and the border (for exports), and from port to end-user (for imports).

Import demand is modeled using the ubiquitous Armington assumption, i.e. goods with the same nomenclature are differentiated by region of origin. This allows for imperfect substitution between domestically produced goods and imported goods. The level of the CES elasticity determines the degree of substitutability across regions of origin. Domestic production is analogously differentiated by region of destination using the constant-elasticity-of-transformation (CET) function. The ability of producers to switch between domestic and foreign markets is determined by the level of the CET elasticity. The model allows for perfect transformation in which case the law-of-one price must hold.

Market equilibrium for domestically produced goods sold domestically is assumed through market clearing prices. By default, the small country assumption is assumed for export and import prices and thus they are exogenous, i.e. export levels do not influence the price received by exporters and import demand does not influence (CIF) import

[1] See Figure 3.1.

prices. The model does allow for implementation of an export demand schedule and an import supply schedule in which case the terms-of-trade would be endogenously determined.

The current version of the model assumes market clearing wages on the labor markets with the possibility of an upward sloping labor supply schedule and sluggish mobility of labor across sectors. Introduction of more labor market segmentation (for example rural versus urban) and some form of wage rigidity could be readily implemented.

Closure of the capital markets depends on the nature of the simulation. In comparative static simulations, capital markets behave similar to labor markets with an upward sloping supply schedule for aggregate capital supply and inter-sectoral capital mobility that depends on a CET transformation elasticity. In dynamic simulations, *new* capital, i.e. that generated by recent investments, is allocated across sectors so as to equalize the rate of return across sectors. *Old* capital remains installed in its original sector unless the sector is in decline. A sector in decline is one in which potential supply, as measured by the capital/output ratio, exceeds *ex post* demand. This can occur from a variety of shocks—for example a negative export demand shock, or a government policy (e.g. a tax on energy) that lowers demand for a specific commodity. If a sector is in decline, it releases its installed capital using an upward sloping supply schedule and its *ex post* return on capital is less than the economy-wide average. *Old* capital in expanding sectors earns the same rate of return as new capital.

The dynamics of MANAGE is composed of three elements. Population and labor stock growth are exogenous—the latter is often equated to the growth of the working age population. The aggregate capital stock grows according to the overall level of saving (enterprises, households, public and foreign), but will also be influenced by the investment price index and the rate of depreciation. The third component relies on productivity assumptions. By default, labor productivity in services is assumed (or calibrated dynamically to achieve a per capita growth target). Labor productivity in other sectors is calculated relative to labor productivity in services using a linear schedule that allows for both multiplicative and additive components. The purpose of this is to calibrate inter-sectoral labor productivity to historical trends (either domestic or international).

Chapter 3

Model specification

3.1 Model dimensions

The coding of the model is relatively independent of the dimensionality of the SAM and other functional dimensions of the data. Table 3.1 provides a listing of the main sets and subsets of the model. The input SAM has a dimensionality of $is \times is$ and most of the remaining sets and subsets derive from is. Sectors have three classifications: a, i, and k, respectively (production) activities, marketed commodities and consumed commodities. In a traditional model the three sets are identical. In the MANAGE model, with its multi-input multi-output production structure, output from activities (a) is combined with imports to supply (or 'produce') commodities (i). This allows, for example, to have multiple activities produce a single commodity (for example electricity), and to have single activities produce multiple commodities (e.g. sugar producing sugar, ethanol, rum and even power). In addition, MANAGE allows for commodities in final demand (indexed by k) to differ from marketed commodities (i). A consumer-based 'make' or transition matrix maps consumed commodities to supplied commodities. This allows for more realistic demand behavior in the context of an energy model. For example, household demand for transportation services can be a combination of demand for fuel and automobiles. If the price of fuel goes up, the combined demand for fuel/autos would decline. It also allows for specific treatment of the demand for fuel and intra-fuel substitutability.

The next sections of the document describe the different block or modules of the model using the rather traditional circular flow scheme of economics, i.e. starting with production and factor incomes, income distribution, demand, trade, and macro closure. At the end, there is a discussion on the model dynamics.

3.2 Production block

The MANAGE production structure relies on a set of nested constant-elasticity-of-substitution (CES) structures.[1] The purpose of the nested CES structure is to replicate the substitution and complementary relations across the various inputs to production. The standard CES nest has intermediate inputs and a value added bundle as a fixed share of output. MANAGE deviates from this standard to allow for capital/energy substitution and complementarity (see figure 1). Production in the dynamic version of the model is based on a vintage structure of capital, indexed by v. In the standard version, there are two vintages—*Old* and *New*, where *New* is capital equipment that is newly installed at the beginning of the period and *Old* capital is capital greater than a year old. The vintage structure impacts model results through two channels. First, it is typically assumed that *Old* capital has lower substitution elasticities than *New* capital. A higher savings rates will lead to a higher share of *New* capital and thus greater overall flexibility. The second channel is through the allocation of capital across sectors. *New* capital is assumed to be perfectly mobile across sectors. *Old* capital is sluggish and released using an upward sloping supply curve. In sectors where demand is declining, the return to capital will be less than the economy-wide average. This is explained in greater detail in the market equilibrium section.

Most of the equations in the production structure are indexed by v, i.e. the capital vintage. The exceptions are those where it is assumed that the further decomposition of a bundle are no longer vintage specific—such as the demand for non-energy intermediate inputs. Each production activity is indexed by a, and is different from the index of produced commodities, i (allowing for the combination of outputs from different activities into a single produced

[1] Some of the key analytical properties of the CES, and its related constant-elasticity-of-transformation (CET) function, are fully described in Appendix A.

Table 3.1: Sets used in model definition

Set	*Description*
is	Full set of SAM Accounts
$aa(is)$	Set of Armington agents—includes all production activities and final demand
$a(aa)$	Set of production activities
$oa(aa)$	Other Armington agents—mostly final demand accounts
$fd(oa)$	Final demand accounts (excludes the trade and transport margin accounts)
$h(oa)$	Household accounts
$f(oa)$	Other final demand accounts
$i(is)$	Commodities
$e(i)$	Energy commodities
$in(i)$	Non-energy commodities
k	Consumed commodities
$nrg(k)$	The energy bundle in consumed commodities
$inst(is)$	Institutions (for transfers)
$fp(is)$	Factors of production
$l(fp)$	Labor categories
$ul(l)$	Unskilled labor types
$sl(k)$	Skilled labor types
v	Vintages (*Old* and *New*)

good, for example electricity). All the equations also allow for a variety of efficiency factors that are used for the dynamic version of the model.

The top level bundle is composed of net output, XPN, and the sector specific natural resource, XNR for those sectors that are subject to a natural resource—typically sectors such as fisheries, forestry and the mining and extractive sectors. Equations (3.1) and (3.2) are derived demands for the natural resource factor and the net output bundle, respectively. The former is aggregated over vintages and the latter is defined by vintage, where XPv represents output by vintage by each activity. The natural resource equation allows for technological change embodied in the λ parameter. Equation (3.3) defines the vintage-specific unit cost, PXv. Almost all CES price equations are based on the dual cost function instead of the aggregate cost or revenue formulation. The unit cost function includes the effects of productivity improvement. Equation (3.4) determines the aggregate unit cost, PX, the weighted average of the vintage-specific unit costs with the weights given by the vintage-specific output levels. Equation (3.5) determines the final market price for output, PP, that is equal to the unit cost augmented by the output tax and/or subsidy, respectively τ^p and τ^s. The equivalence of the tax-adjusted unit cost to the output price is an implication of assuming constant-returns-to-scale technology and perfect competition. The production price can also be adjusted by a volume only tax (or an excise tax), represented by τ^x.

$$XNR_a = \sum_v \alpha^{nr}_{a,v} (\lambda^{nr}_a)^{\sigma^{nr}_{a,v}-1} \left(\frac{PXv_{a,v}}{PNR_a} \right)^{\sigma^{nr}_{a,v}} XPv_{a,v} \tag{3.1}$$

$$XPN_{a,v} = \alpha^{pn}_{a,v} \left(\frac{PXv_{a,v}}{PPN_{a,v}} \right)^{\sigma^{nr}_{a,v}} XPv_{a,v} \tag{3.2}$$

$$PXv_{a,v} = \left[\alpha^{nr}_{a,v} \left(\frac{PNR_a}{\lambda^{nr}_a} \right)^{1-\sigma^{nr}_{a,v}} + \alpha^{pn}_{a,v} \left(PPN_{a,v} \right)^{1-\sigma^{nr}_{a,v}} \right]^{1/(1-\sigma^{nr}_{a,v})} \tag{3.3}$$

$$PX_a = \frac{\sum_v PXv_{a,v} XPv_{a,v}}{XP_a} \tag{3.4}$$

$$PP_a = (1 + \tau^p_a + \tau^s_a) PX_a + \tau^x_a \tag{3.5}$$

Equations (3.6) and (3.7) are derived demands for two bundles, one designated as aggregate value added, *VA*, though it also includes energy demand that is linked to capital, and aggregate intermediate demand, *ND*, a bundle that excludes energy. Both are shares of net output by vintage, *XPN*, with the shares being price sensitive with respect to the ratio of the vintage—specific unit cost, *PPN*, and the component prices, respectively *PVA* and *PND*. The equations allow for technological change embodied in the λ parameters that are allowed to be node-specific. For uniform technological change, the two parameters can be subject to the same percentage change. Equation (3.8) defines the vintage-specific unit cost, *PPN*.

$$ND_a = \sum_v \alpha^{nd}_{a,v} \left(\lambda^n_{a,v}\right)^{\sigma^p_{a,v}-1} \left(\frac{PPN_{a,v}}{PND_a}\right)^{\sigma^p_{a,v}} XPN_{a,v} \tag{3.6}$$

$$VA_a = \alpha^{va}_{a,v} \left(\lambda^v_{a,v}\right)^{\sigma^p_{a,v}-1} \left(\frac{PPN_{a,v}}{PVA_a}\right)^{\sigma^p_{a,v}} XPN_{a,v} \tag{3.7}$$

$$PPN_{a,v} = \left[\alpha^{nd}_{a,v} \left(\frac{PND_a}{\lambda^n_{a,v}}\right)^{1-\sigma^p_{a,v}} + \alpha^{va}_{a,v} \left(\frac{PVA_a}{\lambda^v_{a,v}}\right)^{1-\sigma^p_{a,v}}\right]^{1/(1-\sigma^p_{a,v})} \tag{3.8}$$

The subsequent production nest decomposes the *VA* bundle (value added and energy) into aggregate demand for unskilled labor[2], *LAB1*, demand for land (where appropriate), $Land^d$, and the capital/skill/energy bundle, *KSE*.[3] The key substitution elasticity is given by σ^v, which in a standard model represents capital/labor substitution. An elasticity of 1 implies a Cobb-Douglas technology.[4] Equation (3.9) determines the demand for unskilled labor bundle, *LAB1*. Equation (3.10) determines the demand for land, $Land^d$.[5] Equation (3.11) determines demand for the capital/skill/energy bundle, *KSE*. The final equation in this nest, equation (3.12) defines the unit price of the value added cum energy bundle, *PVA*, by vintage.

$$LAB1_{a,v} = \alpha^{l1}_{a,v} \left(\frac{PVA_{a,v}}{PLAB1_{a,v}}\right)^{\sigma^v_{a,v}} VA_{a,v} \tag{3.9}$$

$$Land^d_a = \sum_v \alpha^{lnd}_{a,v} \left(\lambda^t_a\right)^{\sigma^v_{a,v}-1} \left(\frac{PVA_{a,v}}{PLand^p_a}\right)^{\sigma^v_{a,v}} VA_{a,v} \tag{3.10}$$

$$KSE_{a,v} = \alpha^{kse}_{a,v} \left(\frac{PVA_{a,v}}{PKSE_{a,v}}\right)^{\sigma^v_{a,v}} VA_{a,v} \tag{3.11}$$

$$PVA_{a,v} = \left[\alpha^{l1}_{a,v} PLAB1^{1-\sigma^v_{a,v}}_{a,v} + \alpha^{lnd}_{a,v} \left(\frac{PLand^p{}_a}{\lambda^t_a}\right)^{1-\sigma^v_{a,v}} + \alpha^{kse}_{a,v} PKSE^{1-\sigma^v_{a,v}}_{a,v}\right]^{1/(1-\sigma^v_{a,v})} \tag{3.12}$$

The next nest is a decomposition of the capital/skill/energy bundle, *KSE*, into demand for the capital/skill bundle, *KSK*, and an energy bundle, $XNRG^p$. Equation (3.13) defines the demand for the capital/skill bundle, *KSK*. The substitution elasticity is given by σ^k. Equation (3.14) determines the demand for the energy bundle, $XNRG^p$. Equation (3.15) defines the price of the *KSE* bundle, *PKSE*.

$$KSK_{a,v} = \alpha^{ksk}_{a,v} \left(\frac{PKSE_{a,v}}{PKSK_{a,v}}\right)^{\sigma^k_{a,v}} KSE_{a,v} \tag{3.13}$$

$$XNRG^p_{a,v} = \alpha^e_{a,v} \left(\frac{PKSE_{a,v}}{PNRG^p_{a,v}}\right)^{\sigma^k_{a,v}} KSE_{a,v} \tag{3.14}$$

$$PSKE_{a,v} = \left[\alpha^{ksk}_{a,v} \left(PKSK_{a,v}\right)^{1-\sigma^k_{a,v}} + \alpha^e_{a,v} \left(PNRG^p_{a,v}\right)^{1-\sigma^k_{a,v}}\right]^{1/(1-\sigma^k_{a,v})} \tag{3.15}$$

---

2 The description of skilled and unskilled labor is somewhat arbitrary and left up to the user at runtime. The user defines a subset of labor, *ul*, that will be associated with the *LAB1* bundle, called the unskilled labor bundle. All other labor types will be assigned to the subset, *sl*, and called skilled labor and associated with the *LAB2* bundle. The user can decide to put all labor types, or any combination thereof, in any one of the two subsets.

3 The model differentiates between the producers' cost of factors and the market returns to factors. The former is equal to the market return adjusted for sector and factor-specific *ad valorem* taxes (or subsidies). The relevant variables for the factor cost for producers will have a superscript '*p*'.

4 In the GAMS implementation of the model, a Cobb-Douglas technology is approximated by an elasticity of 1.01.

5 Note that the demand for land is summed over all vintages as there is no subsequent decomposition with respect to vintage.

The capital/skill bundle, KSK, is decomposed into demand for a capital bundle by vintage[6], KB, and a component that represents demand for the skilled labor bundle, $LAB2$. Equations (3.16) through (3.18) determine respectively the sectoral demand for the capital bundle, KB, the skilled labor bundle, $LAB2$, and the price of the capital/skill bundle, $PKSK$.

$$KB_{a,v} = \alpha^{kb}_{a,v} \left(\frac{PKSK_{a,v}}{PKB_{a,v}} \right)^{\sigma^{ks}_{a,v}} KSK_{a,v} \tag{3.16}$$

$$LAB2 = \alpha^{l2}_{a,v} \left(\frac{PKSK_{a,v}}{PLAB2_{a,v}} \right)^{\sigma^{ks}_{a,v}} KSK_{a,v} \tag{3.17}$$

$$PKSK_{a,v} = \left[\alpha^{k}_{a,v} \left(PKB_{a,v} \right)^{1-\sigma^{ks}_{a,v}} + \alpha^{l2}_{a,v} \left(PLAB2_{a,v} \right)^{1-\sigma^{ks}_{a,v}} \right]^{1/(1-\sigma^{ks}_{a,v})} \tag{3.18}$$

The capital bundle, KB, is decomposed into demand for 'mobile' capital by vintage, K^d, and a component that represents demand for sector-specific capital, KF. For the moment, the latter should be used sparingly as it is not fully integrated into the investment module and is simply an exogenous factor. It is normally set to zero in all activities. Equations (3.19) through (3.21) determine respectively the sectoral demand for 'mobile' capital, K^d, sector specific capital, KF, and the price of the capital bundle, PKB. The model allows for a tax/subsidy on the use of the fixed capital, captured by the variable τ^{kf}.

$$K^d_{a,v} = \alpha^{k}_{a,v} \left(\lambda^{k}_{a,v} \right)^{\sigma^{kb}_{a,v}-1} \left(\frac{PKB_{a,v}}{PK^p_{a,v}} \right)^{\sigma^{kb}_{a,v}} KB_{a,v} \tag{3.19}$$

$$KF_a = \sum_v \alpha^{kff}_{a,v} \left(\lambda^{kf}_{a,v} \right)^{\sigma^{kb}_{a,v}-1} \left(\frac{PKB_{a,v}}{(1+\tau^{kf}_a) RKF_a} \right)^{\sigma^{kb}_{a,v}} KB_{a,v} \tag{3.20}$$

$$PKB_{a,v} = \left[\alpha^{k}_{a,v} \left(\frac{PK^p_{a,v}}{\lambda^{k}_{a,v}} \right)^{1-\sigma^{kb}_{a,v}} + \alpha^{kff}_{a,v} \left(\frac{(1+\tau^{kf}_a) RKF_a}{\lambda^{kf}_{a,v}} \right)^{1-\sigma^{kb}_{a,v}} \right]^{1/(1-\sigma^{kb}_{a,v})} \tag{3.21}$$

The two aggregate labor bundles, $LAB1$ and $LAB2$, are broken out into labor demand by skill level, L^d. The inter-labor substitution elasticity is given by σ^{ul} for the unskilled bundle and σ^{sl} for the skilled bundle. Labor demand is added up over all vintages, equation (3.22), and the demand is adjusted by labor productivity factors. Equation (3.23) defines the productivity adjusted cost of aggregate unskilled labor and Equation (3.24) defines the productivity adjusted cost of aggregate skilled labor

$$L^d_{a,l} = \begin{cases} \displaystyle\sum_v \alpha^{l}_{a,l,v} \left(\lambda^{l}_{a,l} \right)^{\sigma^{ul}_{a,v}-1} \left(\frac{PLAB1_{a,v}}{W^p_{a,l}} \right)^{\sigma^{ul}_{a,v}} LAB1_{a,v} & \text{if } l \in ul \\ \displaystyle\sum_v \alpha^{l}_{a,l,v} \left(\lambda^{l}_{a,l} \right)^{\sigma^{sl}_{a,v}-1} \left(\frac{PLAB2_{a,v}}{W^p_{a,l}} \right)^{\sigma^{sl}_{a,v}} LAB2_{a,v} & \text{if } l \in sl \end{cases} \tag{3.22}$$

$$PLAB1_{a,v} = \left[\sum_{ul} \alpha^{l}_{a,ul,v} \left(\frac{W^p_{a,ul}}{\lambda^{l}_{a,ul}} \right)^{1-\sigma^{ul}_{a,v}} \right]^{1/(1-\sigma^{ul}_{a,v})} \tag{3.23}$$

$$PLAB2_{a,v} = \left[\sum_{sl} \alpha^{l}_{a,sl,v} \left(\frac{W^p_{a,sl}}{\lambda^{l}_{a,sl}} \right)^{1-\sigma^{sl}_{a,v}} \right]^{1/(1-\sigma^{sl}_{a,v})} \tag{3.24}$$

The final two nodes in the production nest decompose respectively aggregate demand for non-fuel intermediate goods, ND, and the energy bundle, $XNRG^p$. Normally we assume a standard Leontief technology for the former, but the model allows for substitution across non-energy inputs.[7] Equation (3.25) determines the demand for the (Armington) intermediate demand for non-fuel inputs, XA_{in}, with the substitution elasticity given by σ^n. The relevant price is the Armington price, PA, which in the current version of the model is assumed to be uniform

6 [NEW:26-Mar-2018] A new factor of production has been introduced that represents a sector-specific capital. For the moment, it is not vintage specific. It is bundled with 'mobile capital.

7 The model would need to be adapted to allow for specific substitution across selected inputs—for example across different animal feed products.

economy-wide, i.e. all agents have identical preferences for domestic versus imported goods, but that does allow for different end-user prices using a tax/subsidy instrument, and as well a carbon tax that can similarly be user-specific. This is detailed further below. The variable *PAF* represents the end-user price of the Armington good that is inclusive of the normal indirect tax and potentially of the carbon tax.[8] Equation (3.26) provides the price of the aggregate *ND* bundle, where the set *nnrg*, covers all non-energy intermediate inputs. Equations (3.27) and (3.28) perform a similar decomposition for the energy bundle, determining the demand for specific energy carriers (indexed by e).[9] The inter-energy substitution elasticity is given by σ^e.[10] These equations also include energy efficiency parameters that are specific to the sector of activity, energy carrier and vintage.

$$XA_{in,a} = \alpha^{io}_{in,a}\left(\frac{PND_a}{\chi^{PA}_{in} PAF_{in,a}}\right)^{\sigma^n_a} ND_a \tag{3.25}$$

$$PND_a = \left[\sum_{in \in nnrg} \alpha^{io}_{in,a}\left(\chi^{PA}_{in} PAF_{in,a}\right)^{1-\sigma^n_a}\right]^{1/(1-\sigma^n_a)} \tag{3.26}$$

$$XA^d_{e,a} = \sum_v \alpha^{ep}_{e,a,v}\left(\lambda^e_{e,a,v}\right)^{\sigma^e_{a,v}-1}\left(\frac{PNRG^p_{a,v}}{PAF_{e,a}}\right)^{\sigma^e_{a,v}} XNRG^p_{a,v} \tag{3.27}$$

$$PNRG^p_{a,v} = \left[\sum_e \alpha^{ep}_{e,a,v}\chi^{PA}_e\left(\frac{PAF_{e,a}}{\lambda^e_{e,a,v}}\right)^{1-\sigma^e_{a,v}}\right]^{1/(1-\sigma^e_{a,v})} \tag{3.28}$$

This ends the description of the production structure, though there is a further decomposition of XA_{in}, i.e. the non-fuels intermediate Armington demand, and the energy bundle into imported and domestic components.

The next section of the supply block describes the multi-input/multi-output specification of the model. The multi-input part of the model allows for the aggregation of multiple activities into a single marketed good. This specification is useful for example in the electricity sector where the final output is the combination of multiple electricity production activities—hydro, coal-powered, renewables, etc. The specification allows for various levels of substitution—including the law-of-one-price that assumes that output from all activities is homogeneous and the unit cost of production is identical across activities. The multi-output part of the model allows for joint production, for example producing ethanol and DDGS from ethanol production.

Activity a can therefore produce a suite of commodities indexed by i, hence an output at this level is indexed by both a and i, $X_{a,i}$. This is implemented using a CET structure with the possibility of infinite transformation. Equation (3.29) defines the supply of $X_{a,i}$ emanating from activity a (or XP_a), where the law-of-one- price holds in the case of infinite transformation. Equation (3.30) represents the zero profit condition, or the revenue balance for the multi-output production function. Again, the χ^{us} parameters are used as normalization factors so that the price variables are initialized at 1, but whenever used, need to be scaled by the relevant χ to preserve the correct accounting.

$$\begin{cases} X_{a,i} = \gamma^u_{a,i}\left(\dfrac{\chi^{us}_{a,i} P_{a,i}}{PP_a}\right)^{\omega^p_a} XP_a & \text{if } \omega^p_a \neq \infty \\ \chi^{us}_{a,i} P_{a,i} = PP_a & \text{if } \omega^p_a = \infty \end{cases} \tag{3.29}$$

$$PP_a XP_a = \sum_{\left\{i | \chi^{us}_{a,i} \neq 0\right\}} \chi^{us}_{a,i} P_{a,i} X_{a,i} \tag{3.30}$$

In the next step, multiple streams of output can be combined into a single supplied commodity, XS_i, with a CES-aggregator. The specification allows for homogeneous goods, for example electricity—in which case the cost of each component must be equal.[11] Equation (3.31) determines the demand for produced commodity X. In the case of a finite elasticity it is a CES formulation. With an infinite substitution elasticity, the law-of-one price must hold, i.e. the producer price of each component must be equalized in efficiency units. Equation (3.32) determines

8 To enhance the numerical properties of the model, the basic Armington price is normalized to 1, even in the case of correct price/volume splits. The parameter χ is used to convert the normalized price to the correct level, for example, dollar (or local currency) per liter, or, in terms of any other energy volume used by the model, for example million or thousand tons of oil equivalent (MTOE or TOE).

9 There is a slight difference in the treatment of the price normalization factor for the energy bundles. The share parameters α^{ep} incorporate the price normalization factor directly.

10 The energy decomposition herein is described with a single nest. More complex substitution patterns can be specified by adding additional nests—for example having an electricity bundle, a liquid fuels bundle, etc.

11 Subject perhaps to an efficiency differential.

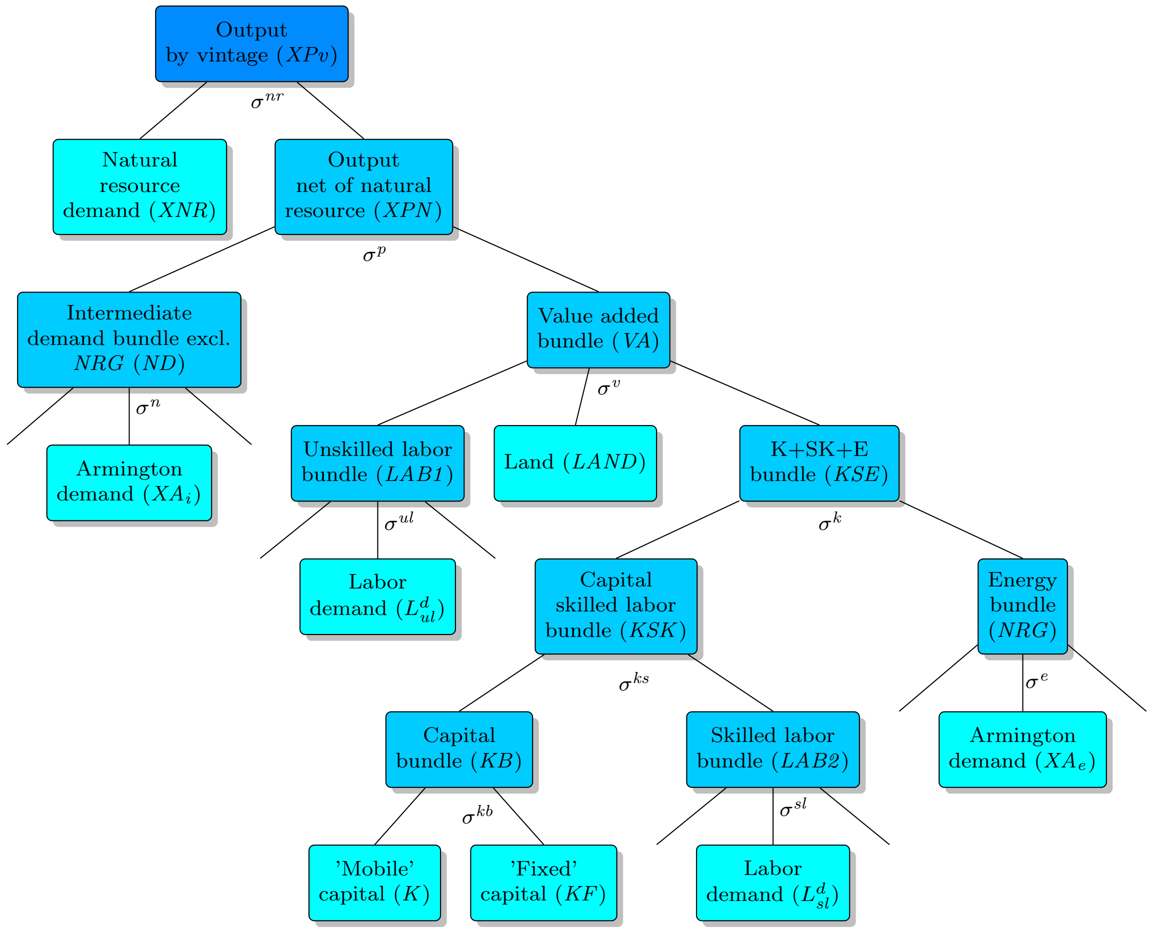

Figure 3.1: **Production nest**

the equilibrium condition in the form of the cost function expression, or (primal) volume equality if the law-of-one price holds.[12] The formulas allow for shifts in preferences via the λ^s parameter.[13] One possible preference shift can emerge from a cost neutral shift in the preference for one component of the CES function (see Appendix A).

$$\begin{cases} X_{a,i} = \alpha^u_{a,i} \left(\lambda^{ud}_{a,i}\right)^{\sigma^s_i - 1} \left(\dfrac{PS_i}{\chi^{ud}_{a,i} P_{a,i}}\right)^{\sigma^s_i} XS_i & \text{if } \sigma^s_i \neq \infty \\ \chi^{ud}_{a,i} P_{a,i} = PS_i & \text{if } \sigma^s_i = \infty \end{cases} \tag{3.31}$$

$$\begin{cases} PS_i = \left[\displaystyle\sum_a \alpha^u_{a,i} \left(\dfrac{\chi^{ud}_{a,i} P_{a,i}}{\lambda^{ud}_{a,i}}\right)^{1-\sigma^s_i}\right]^{1/(1-\sigma^s_i)} & \text{if } \sigma^s_i \neq \infty \\ XS_i = \displaystyle\sum_a X_{a,i} & \text{if } \sigma^s_i = \infty \end{cases} \tag{3.32}$$

3.3 Income block

In the current version of the model, households derive their income from factor payments.[14] Equation (3.33) defines gross profits, i.e. total capital remuneration including net of taxes returns to sector specific capital. Equation (3.34)

12 The model explicitly assumes supply/demand equilibrium for the variable X, thus the superscripts d and s are suppressed as well as the equilibrium condition.

13 The preference shift parameter is new to Version 2.0c.

14 The latest version of the model includes enterprises and a variety of domestic and international transfers.

defines gross income for household h, YH. Each term represents the share of total after-tax remuneration accruing to household 'h'. The first term on the right-hand side includes aggregate wage payments by skill level, where the matrix of coefficients χ^l distributes the payment from skill level l to household h. The second term represents the after tax distribution of land income. The third term represents aggregate payment to capital services and is distributed to households using the coefficient matrix χ^k. The fourth term represents the after tax distribution of natural resource income. The fifth term is the sum of all transfers from institutions to households. Equation (3.35) defines household direct tax payment, Tax^h—it is a linear line with potentially a non-zero intercept. The variable τ^d represents the marginal tax rate for household h and ϕ^d is the intercept of the tax schedule. Both tax parameters are multiplied by an adjustment factor that is set to 1 in the base year. Depending on the closure rule, one of the adjustment factors may be endogenous to achieve some target. For example, one closure may fix the fiscal deficit of the government. One way to achieve the fiscal target is to endogenize the tax schedule—using either a uniform shift on the marginal tax rate or a uniform shift on the intercept (assuming the base year intercept is different from zero for at least one household). Equation (3.36) determines the average savings propensity out of after-tax income, aps. It is simply constant relative to the bench-mark level with the possibility of a uniform shifter depending on the closure rule. For example, to achieve a targeted level of investment, the parameter χ^s may be endogenized. Equation (3.37) specifies the total level of transfers from households—it is simply a share of total income. Equation (3.38) determines available household income for purchases of goods and services, YF. Finally, equation (3.39) determines household savings, S^h.

$$KAPY = \sum_a \sum_v PK_{a,v} K^d_{a,v} + \sum_a RKF^n_a KF_a \tag{3.33}$$

$$\begin{aligned} YH_h \quad = \quad & \left(1-\kappa^l_l\right) \sum_l \chi^l_{h,l} \sum_a W_{a,l} L^d_{a,l} + \left(1-\kappa^t\right) \chi^t_h \sum_a PLand_a XLand^d_a \\ + \quad & \left(1-\kappa^k\right) \chi^k_h KAPY + \left(1-\kappa^{nr}\right) \chi^{nr}_h \sum_a PNRS_a XNRS^d_a \\ + \quad & \sum_{inst} Transfers_{h,inst} \end{aligned} \tag{3.34}$$

$$Tax^h_h = \chi^m \tau^d_h YH_h + \chi^a \phi^d_h \tag{3.35}$$

$$aps_h = \chi^s aps_{h,0} \tag{3.36}$$

$$totTr_h = \chi^{totTr}_h YH_h \tag{3.37}$$

$$YF_h = (1 - aps_h) \left(YH_h - totTr_h - Tax^h_h \right) \tag{3.38}$$

$$S^h_h = aps_h \left(YH_h - totTr_h - Tax^h_h \right) \tag{3.39}$$

Government revenues have been divided into separate streams in order to simplify the notation: subsidies and taxes on production ("ptx"), value added taxes ("otx"), import and export taxes ("mtx")[15], agent-specific sales taxes and subsidies ("atx"), factor taxes and subsidies ("vtx"), and direct taxes ("dtx").[16] The production tax stream includes taxes/subsidies on the use of the sector specific factor. Equations 3.40 through 3.46 describe the different tax streams. (N.B. not all variables have been defined yet.) Equation (3.45) defines the level of carbon tax revenue. The separate variables and parameters will be described in the emissions module below.

$$YG_{ptx} = \sum_a \left(\tau^p_a + \tau^s_a\right) PX_a XP_a + \sum_a \tau^{kf}_a RKF_a KF_a \tag{3.40}$$

$$YG_{otx} = \sum_i \left[\tau^{va,d}_i \left(\chi^{PD}_i PD_i + PMARG_i \zeta^d_i \right) XD_i + \tau^{va,m}_i PM_i XM_i \right] \tag{3.41}$$

$$YG_{mtx} = \sum_i \left(PMD_i - ER \cdot PWM_i\right) \left(XM_i + STB^m_i\right) + \sum_i \tau^e_i \chi^{PE}_i PE_i XE_i \tag{3.42}$$

$$YG_{atx} = \sum_{aa} \sum_i \left(\tau^a_{i,aa} + \varsigma^a_{i,aa}\right) \chi^{PA}_i PA_i XA_{i,aa} \tag{3.43}$$

15 The formulation for import tariff revenues allows for imperfect price pass-through in which case the government absorbs the difference between the border price and the (tariff-inclusive) end-user price.

16 Agent-specific subsidies were added with version 2.0c.

$$\begin{aligned} YG_{vtx} &= \sum_l \sum_a \tau^l_{a,l} W_{a,l} L^d_{a,l} + \sum_a \tau^t_a PLand_a Land^d_a \\ &+ \sum_v \sum_a \tau^k_{a,v} PK_{a,v} K^d_{a,v} + \sum_a \tau^{nr}_a PNRS_a XNRS^d_a \end{aligned} \tag{3.44}$$

$$YG_{ctx} = \sum_{em} \sum_{aa} \sum_i \tau^{em}_{em} \chi^{Emi}_{em,i,aa} \rho^{Emi}_{em,i,aa} \varphi^{Emi}_{em,i,aa} XA_{i,aa} \tag{3.45}$$

$$\begin{aligned} YG_{dtx} &= \sum_h Tax^h_h + \sum_{entr} Tax^{entr}_{entr} + Tax^{gov} \\ &+ \kappa^l_l \sum_l \sum_a W_{a,l} L^d_{a,l} + \kappa^t \sum_a PLand_a XLand^d_a \\ &+ \kappa^k KAPY + \kappa^{nr} \sum_a PNRS_a XNRS^d_a \\ &+ \textstyle\sum_a \kappa^{kf}_a RKF_a KF_a \end{aligned} \tag{3.46}$$

The next block of equations determines domestic closure for the government accounts. Equation (3.47) defines the total revenues for the government. There are three components—aggregate taxes, transfers from domestic and foreign institutions and the residual share of profits.[17] Equation (3.48) determines direct taxes on government operations [!!!need to identify]. Equation (3.49) determines total public transfers as a share of government revenues. Nominal government saving, S^g, is the difference between government revenues (summed over the index g) and the total of government current expenditures—direct taxes, transfers and public expenditures on goods and services, YF_{gov}, see equation (3.50). Equation (3.51) defines real government savings, RS^g, using a yet-to-be-defined economy-wide deflator, $PNUM$.

$$YGOV = \sum_g YG_g + \sum_{inst} Transfers_{gov,inst} + KAPY \left(1 - \sum_h \chi^k_h - \sum_{entr} \chi^k_{entr}\right) \tag{3.47}$$

$$Tax^{gov} = \kappa^{gov} YGOV \tag{3.48}$$

$$totTr_{gov} = \chi^{totTr}_{gov} YGOV \tag{3.49}$$

$$S^g = \sum_g YG_g - Tax^{gov} - totTr_{gov} - YF_{gov} \tag{3.50}$$

$$RS^g = S^g / PNUM \tag{3.51}$$

The next set of equations relates to the enterprise accounts that are essentially pass-through accounts with minimal behavior. Equation (3.52) defines enterprise incomes—a share of total profits plus transfers. Equation (3.53) determines direct taxes on enterprises. Equation (3.54) determines enterprise savings that feeds into the investment stream. And equation (3.55) determines aggregate enterprise transfers as a residual after tax and savings.

$$YEntr_{entr} = \chi^k_{entr} KAPY + \sum_{inst} Transfers_{entr,inst} \tag{3.52}$$

$$Tax^{entr}_{entr} = \kappa^{entr} YEntr_{entr} \tag{3.53}$$

$$S^{entr}_{entr} = \alpha^s_{entr} \left(YEntr_{entr} - Tax^{entr}_{entr}\right) \tag{3.54}$$

$$totTr_{entr} = YEntr_{entr} - Tax^{entr}_{entr} - S^{entr}_{entr} \tag{3.55}$$

The final equations in the income block allocate transfers across institutions—handling separately the domestic and incoming foreign transfers. Equation (3.56) allocates the total amount allocated for transfers by domestic agents (indexed by *dinst*) using a matrix of share coefficients, χ^{tr}. Foreign transfers are exogenous and fixed in foreign currency terms, equation (3.57). They are converted to domestic currency units by the exchange rate, ER. The final equation is the ubiquitous investment equals savings equation (3.58). Nominal investment, YF_{Inv}, is equal to the sum of savings from households, enterprises, government and from abroad, adjusted for the value of stock building.[18]

$$Transfers_{inst,dinst} = \chi^{tr}_{inst,dinst} totTr_{dinst} \tag{3.56}$$

[17] An alternative would be to identify public profit shares by sector.

[18] N.B. The value of imports of stocks is the tariff-inclusive domestic price-excluding margin and other indirect tax wedges.

$$Transfers_{inst,row} = ER\,Transfers^{ROW}_{inst,row} \tag{3.57}$$

$$\begin{aligned} YF_{Inv} &= \sum_h S^h_h + \sum_{entr} S^{entr}_{entr} + S^g + ER \cdot S^f \\ &- \sum_i \left[\chi^{PS}_i PS_i STB^d_i + PM^d_i STB^m_i\right] \end{aligned} \tag{3.58}$$

3.4 Demand block

The household demand system is derived from the constant-difference-in-elasticity (CDE) utility function—the standard demand system in the GTAP model. It is based on an implicitly additive utility function and can collapse to a CES utility function under certain conditions. It provides a fair amount of flexibility and is typically calibrated to external estimates of both income and own-price elasticities. The function and some of its key characteristics are derived more fully in Appendix B.

To simplify implementation, Equation (3.59) defines an auxiliary variable, *ZCons*, that simplifies the remaining expressions for the CDE. Equation (3.60) defines the budget share, s^{kf}, for household h for consumer good k, and is based on the auxiliary variable *ZCons*. Equation (3.61) represents aggregate demand for good k by household h, XKF, using the standard expression for budget shares.[19] The model allows for a different classification of consumer goods and producer goods. A 'make' matrix—described below—converts consumer goods to producer goods. The utility function is based on personal utility. Equation (3.62) determines household utility (on a per capita basis), u, and users the auxiliary variable *ZCons* to simplify the expression of the equation. Equation (3.63) defines the household specific consumer price index, PF, and is a weighted sum of consumer goods prices.

$$ZCons_{h,k} = \alpha^{kf}_{h,k} b_{h,k} u_h^{e_{h,k} b_{h,k}} \left(\frac{PKF_{h,k}}{YF_h Pop_h}\right)^{b_{h,k}} \tag{3.59}$$

$$s^{kf}_{h,k} = \frac{ZCons_{h,k}}{\sum_{k'} ZCons_{h,k'}} \tag{3.60}$$

$$XKF_{h,k} = \frac{s^{kf}_{h,k}}{PKF_{h,k}} YF_h \tag{3.61}$$

$$\sum_k \frac{ZCons_{h,k}}{b_{h,k}} \equiv 1 \tag{3.62}$$

$$PF_h = \sum_k s^{kf}_{h,k} PKF_{h,k} \tag{3.63}$$

The next set of equations decomposes household demand defined as consumed goods into supplied (or more accurately, Armington) goods. A transition matrix approach is used where each consumed good is composed of one or more supplied goods and combined using a CES aggregator.[20] Each consumer good could also have its own energy bundle—with different demand shares across energy.[21] A nested CES structure is deployed to convert consumed goods (k) to supplied goods (i) across households (h). The top nest decomposes demand for good k into a non-energy bundle, XKF^{NNRG}, and an energy bundle, XKF^{NRG}—see respectively equations (3.64) and (3.65). Equation (3.66) defines the price of consumed good k for household h, PKF.

$$XKF^{NNRG}_{h,k} = \alpha^{NNRG}_{h,k} XKF_{h,k} \left(\frac{PKF_{h,k}}{PKF^{NNRG}_{h,k}}\right)^{\sigma^a_{h,k}} \tag{3.64}$$

[19] N.B. That population drops out of each side of the equation since both XKF and YF are defined in aggregate terms, not per capita.

[20] In most cases, the transition matrix is almost diagonal—i.e. each consumed good is mapped to a single produced good. The exceptions are the energy goods that are aggregated together into a single energy bundle that allows for substitution across the different energy carriers. A satellite dataset would be needed to calibrate a more complex transition matrix.

[21] For example, a transportation bundle is likely to be dominated by liquid fuel demand, whereas demand for heat is likely to be dominated by electricity and natural gas.

$$XKF_{h,k}^{NRG} = \alpha_{h,k}^{NRG} XKF_{h,k} \left(\frac{PKF_{h,k}}{PKF_{h,k}^{NRG}}\right)^{\sigma_{h,k}^{a}} \tag{3.65}$$

$$PKF_{h,k} = \left[\alpha_{h,k}^{NNRG}\left(PKF_{h,k}^{NNRG}\right)^{1-\sigma_{h,k}^{a}} + \alpha_{h,k}^{NRG}\left(PKF_{h,k}^{NRG}\right)^{1-\sigma_{h,k}^{a}}\right]^{1/(1-\sigma_{h,k}^{a})} \tag{3.66}$$

The next step decomposes the aggregate non-energy bundle into Armington demand for produced (and imported) goods indexed by in. The demand is summed across all k consumer goods—equation (3.67). Equation (3.68) performs the same function for the energy bundle, but also allows for energy efficiency improvement though the λ parameter that is specific to each household (h), each consumer good (k) and for each energy carrier (e). Thus one could assume more rapid improvement in energy efficiency in household demand for residential energy use than in transportation demand, or vice versa. Finally, equations (3.69) and (3.70) determine the price of the respective non energy and energy bundles for each agent and each consumed commodity.

$$XA_{in,h} = \sum_{k} \alpha_{h,in,k}^{f} XKF_{h,k}^{NNRG} \left(\frac{PKF_{h,k}^{NNRG}}{\chi_{in}^{PA} PAF_{in,h}}\right)^{\sigma_{h,k}^{ca}} \tag{3.67}$$

$$XA_{e,h} = \sum_{k} \alpha_{h,e,k}^{f} \frac{XKF_{h,k}^{NRG}}{\lambda_{h,e,k}^{eh}} \left(\frac{\lambda_{h,e,k}^{eh} PKF_{h,k}^{NRG}}{\chi_{e}^{PA} PAF_{e,h}}\right)^{\sigma_{h,k}^{ce}} \tag{3.68}$$

$$PKF_{h,k}^{NNRG} = \left[\sum_{in} \alpha_{h,in,k}^{f}\left(\chi_{in}^{PA} PAF_{in,h}\right)^{1-\sigma_{h,k}^{ca}}\right]^{1/(1-\sigma_{h,k}^{ca})} \tag{3.69}$$

$$PKF_{h,k}^{NRG} = \left[\sum_{e} \alpha_{h,e,k}^{f}\left(\frac{\chi_{e}^{PA} PAF_{e,h}}{\lambda_{h,e,k}^{eh}}\right)^{1-\sigma_{h,k}^{ce}}\right]^{1/(1-\sigma_{h,k}^{ce})} \tag{3.70}$$

Demand decomposition of the other final demand accounts (indexed by f) uses a similar transition matrix/nested CES approach as household demand. Equations (3.71) and (3.72) determine respectively demand for the non-energy and energy bundles for each other final demand agent, f, where XF, the aggregate volume of expenditure, is determined using specific closure rules. Equation (3.73) reflects the standard price aggregation function determining PF.

$$XF_{f}^{NNRG} = \alpha_{f}^{NNRG} XF_{f} \left(\frac{PF_{f}}{PF_{f}^{NNRG}}\right)^{\sigma_{f}^{f}} \tag{3.71}$$

$$XF_{f}^{NRG} = \alpha_{f}^{NRG} XF_{f} \left(\frac{PF_{f}}{PF_{f}^{NRG}}\right)^{\sigma_{f}^{f}} \tag{3.72}$$

$$PF_{f} = \left[\alpha_{f}^{NNRG}\left(PF_{f}^{NNRG}\right)^{1-\sigma_{f}^{f}} + \alpha_{f}^{NRG}\left(PF_{f}^{NRG}\right)^{1-\sigma_{f}^{f}}\right]^{1/(1-\sigma_{f}^{f})} \tag{3.73}$$

The next block of equations determine the demand for Armington goods for agents f. Equations (3.74) and (3.75) determine respectively the demand for Armington goods—partitioned into non-energy and energy commodities. And equations (3.76) and (3.77) determine respectively the price of the non-energy and energy bundles.

$$XA_{in,f} = \alpha_{f,in}^{f} XF_{f}^{NNRG} \left(\frac{PF_{f}^{NNRG}}{\chi_{in}^{PA} PAF_{in,f}}\right)^{\sigma_{f}^{fa}} \tag{3.74}$$

$$XA_{e,f} = \alpha_{f,e}^{f} \frac{XF_{f}^{NRG}}{\lambda_{f,e}^{ef}} \left(\frac{\lambda_{f,e}^{ef} PF_{f}^{NRG}}{\chi_{e}^{PA} PAF_{e,f}}\right)^{\sigma_{h,k}^{fe}} \tag{3.75}$$

$$PF_{f}^{NNRG} = \left[\sum_{in} \alpha_{f,in}^{f}\left(\chi_{in}^{PA} PAF_{in,f}\right)^{1-\sigma_{f}^{fa}}\right]^{1/(1-\sigma_{f}^{fa})} \tag{3.76}$$

$$PF_{f}^{NRG} = \left[\sum_{e} \alpha_{f,e}^{f}\left(\frac{\chi_{e}^{PA} PAF_{e,f}}{\lambda_{f,e}^{ef}}\right)^{1-\sigma_{f}^{fe}}\right]^{1/(1-\sigma_{f}^{fe})} \tag{3.77}$$

The final set of equations in the demand block refer to the demand for goods and services for the trade and transport margins. The model has three different nodes for margins: 1) from farm or factory gate to domestic markets ('d'); 2) from farm or factor gate to the domestic border for exports ('e'); and 3) from the border to domestic markets for imports ('m'). Though the wedges are different for each of the three nodes—the unit cost of the margin is the same across all nodes (i.e. the 'production' function of the trade and transport margins is identical across all nodes). For each commodity i, the unit cost of the margin is $PMARG_i$ irrespective of the node. Equation (3.78) expresses the demand for margin services for good j across all three nodes, $XMARG$. This demand generates demand for 'Armington' goods, where a CES cost function is assumed—equation (3.79).[22] The cost of the trade margins is reflected in the price, $PMARG$, and determined in equation (3.80). The differential cost structure for margins across goods will reflect the different energy intensities as reflected in the cost shares (and the indirect demand for energy through the input output table).

$$XMARG_j = \zeta_j^d XD_j + \zeta_j^e XE_j + \zeta_j^m XM_i j \tag{3.78}$$

$$XA_{i,j} = \alpha_{i,j}^{mrg} XMARG_j \left(\frac{PMARG_j}{\chi_i^{PA} PA_i} \right)^{\sigma_j^{mg}} \tag{3.79}$$

$$PMARG_j = \left[\sum_i \alpha_{i,j}^{mrg} \left(\chi_i^{PA} PA_i \right)^{1-\sigma_j^{mg}} \right]^{1/(1-\sigma_j^{mg})} \tag{3.80}$$

The final equation in the demand block, equation (3.81), reflects the price/volume split for the final demand agents. What the equation 'determines' will be reflected by specific closure rules to be discussed below.

$$YF_{oa} = PF_{oa} XF_{oa} \tag{3.81}$$

3.5 Trade block

3.5.1 Demand for domestic and imported goods

The equations above have determined completely the so-called Armington demand for goods across all agents, XA, that include activities (a), private or consumer demand (h), other final demand (f) and for margin delivery (j). The union of these three sets is the set aa. All Armington agents are assumed to have the same preference function for domestic and imported goods.[23] It is also assumed that the Armington good, for each commodity i, is homogeneous across agents, and can therefore be aggregated in volume terms.

Equations (3.82) through (3.84) define the different import prices used in the model. The CIF price is reflected in the variable PWM, that is multiplied by the exchange rate (ER). The domestic price, PM^d, is then the CIF price, in local currency, adjusted by the import tariff—equation (3.82). The model allows for imperfect price pass-through as reflected in the parameter π. The default value for π is 1, i.e. perfect price pass-through.[24] The tariff revenue equation, (3.42), has been modified under the assumption that the government subsidizes imports in the event of less than perfect price pass-through. The import price is then augmented by the domestic trade and transport margins to become PM—equation (3.83). Finally, a sales or value added tax is added to the import price to become the agents' price of imports, PM^a—equation (3.84). It is this price that will be reflected in the substitution decision between imports and domestic goods. Equation (3.85) determines the agents' price of domestically produced goods, PD^a, equal to the producer price, PD, augmented by trade and transport margins and the sales or value added tax.

$$PM_i^d = ER(1+\tau_i^m) \left[\pi_i PWM_i + (1-\pi_i) PWM_{i,0} \right] \tag{3.82}$$

$$PM_i = PM_i^d + PMARG_i \zeta_i^m \tag{3.83}$$

$$\chi_i^{PMa} PM_i^a = PM_i (1+\tau_i^{va,m}) \tag{3.84}$$

$$\chi_i^{PDa} PD_i^a = \left(PD_i + PMARG_i \zeta_i^d \right) \left(1+\tau_i^{va,d} \right) \tag{3.85}$$

22 Margin services are not associated with specific indirect taxes—though this could be changed in future revisions.

23 Some national SAM's allow for the decomposition of Armington demand into its domestic and import component by agent.

24 [NEW] The price pass-through parameter was introduced into the model 25-May-2018.

Equation (3.86) defines aggregate Armington demand, XAT. It is the sum across all agents of their Armington demand. As described above, the decomposition of the Armington aggregate, XAT, is done at the national level. Aggregate national demand for domestic goods, XD, is then a fraction of XAT, with the fraction sensitive to the relative price of domestic goods, PD^a, to the Armington good, PA—as shown in equation (3.87)). The key parameter, known as the Armington substitution elasticity, is σ^m. Equation (3.88) determines the demand for imports, XM. Equation (3.89) defines the aggregate (or national) price of the aggregate Armington good, PA. Equation (3.90) defines the end-user price of the Armington good. In the absence of a carbon tax, the end-user price is equal to the basic Armington price adjusted by a user-specific sales tax, τ^a, and/or subsidy, ς^a. The carbon tax is composed of several parts. The parameter ρ^{em} measures the quantity of emissions per unit of absorption. It can be adjusted over time by the factor χ^{em}. The carbon tax is given by τ^{em} and is in local currency per unit of emission. The parameter φ^{em} allows for variable participation rates across end-users. If it is set to 1, the participant is subject to the full tax. If it is 0, the end-user is exempt from the carbon tax. The carbon tax is scaled by the Armington price scale factor for accounting purposes.

$$XAT_i = \sum_{aa} XA_{i,aa} \tag{3.86}$$

$$XD_i^d = \alpha_i^d \left(\frac{\gamma_i^d PA_i}{PD_i^a}\right)^{\sigma_i^m} XAT_i \tag{3.87}$$

$$XM_i = \alpha_i^m \left(\frac{\gamma_i^m PA_i}{PM_i^a}\right)^{\sigma_i^m} XAT_i \tag{3.88}$$

$$PA_i = \left[\alpha_i^d \left(\gamma_i^d PD_i^a\right)^{1-\sigma_i^m} + \alpha_i^m \left(\gamma_i^m PM_i^a\right)^{1-\sigma_i^m}\right]^{1/(1-\sigma_i^m)} \tag{3.89}$$

$$PAF_{i,aa} = \left(1 + \tau_{i,aa}^a + \varsigma_{i,aa}^a\right) PA_i + \frac{1}{\chi_i^{PA}} \sum_{em} \tau_{em}^{em} \chi_{emi,aa}^{em} \rho_{em,i,aa}^{em} \varphi_{em,i,aa}^{em} \tag{3.90}$$

3.5.2 Export supply

Analogous to the Armington specification described above, the model allows for imperfect transformation of output across markets of destination—domestic and for export. A constant-elasticity-of-transformation (CET) structure is implemented. Domestic output is allocated between the domestic market and the export market. Infinite transformation is allowed in which case the CET first order conditions are replaced by the law-of-one-price.

Equations (3.91) and (3.92) represent the derived supply for domestic, XD^s, and export, XE^s, markets respectively. With finite transformation, these conditions are the standard CET first order conditions based on supply (less stock building). With perfect transformation, each is replaced with the law-of-one-price whereby the domestic, PD, and export, PE, producer prices are set equal to the aggregate supply price, PS. Equation (3.93) represents the market equilibrium for supply. With perfect transformation (where all prices are uniform) domestic supply is equal to the sum of supply to the various markets.

$$\begin{cases} \gamma_i^{XD} PD_i = PS_i & \text{if } \sigma_i^x = \infty \\ XD_i^s = \gamma_i^d \left(\frac{\gamma_i^{XD} PD_i}{PS_i}\right)^{\sigma_i^x} [XS_i - STB_i] & \text{if } \sigma_i^x \neq \infty \end{cases} \tag{3.91}$$

$$\begin{cases} \gamma_i^{XE} PE_i = PS_i & \text{if } \sigma_i^x = \infty \\ XE_i^s = \gamma_i^e \left(\frac{\gamma_i^{XE} PE_i}{PS_i}\right)^{\sigma_i^x} [XS_i - STB_i] & \text{if } \sigma_i^x \neq \infty \end{cases} \tag{3.92}$$

$$\begin{cases} XS_i = XD_i^s + XE_i^s + STB_i & \text{if } \sigma_i^x = \infty \\ PS_i = \left[\gamma_i^d \left(\gamma_i^{XD} PD_i\right)^{1+\sigma_i^x} + \gamma_i^e \left(\gamma_i^{XE} PE_i\right)^{1+\sigma_i^x}\right]^{1/(1+\sigma_i^x)} & \text{if } \sigma_i^x \neq \infty \end{cases} \tag{3.93}$$

Equation (3.94) defines the domestic producer price of exports, PE. The FOB price of exports is PWE, and is multiplied by ER to convert into local currency terms. There are two wedges that affect the domestic producer price of exports—an export tax and the domestic trade and transport margin. The model allows for both a downward sloping export demand curve and an upward sloping import supply curve. Equation (3.95) reflects the export demand curve, with the possibility of an infinite demand elasticity in which case the FOB price of exports is fully exogenous. The variable PWE^{RoW} represents a price index of competitive exports and is normally exogenous. If the export demand elasticity is finite, export demand declines as the price of home country exports increases. Equation (3.96) provides an analogous treatment of import supply. Import supply would tend to increase with a rise in the price of imports into the home country (for example with a reduction in import tariffs). With an infinite supply elasticity, the price of imports is fixed at PWM^{RoW}.

$$\chi_i^{PE} PE_i\left(1+\tau_i^e\right) + PMARG_i \zeta_i^e = ER \chi_i^{PWE} PWE_i \tag{3.94}$$

$$\begin{cases} PWE_i = \gamma_i^{PWE} PWE_i^{RoW} & \text{if } \sigma_i^x = \infty \\ XE_i^d = \chi_i^e \left(\dfrac{\gamma_i^{PWE} PWE_i^{RoW}}{PWE_i}\right)^{\eta_i^e} & \text{if } \eta_i^e \neq \infty \end{cases} \tag{3.95}$$

$$\begin{cases} PWM_i = PWM_i^{RoW} & \text{if } \omega_i^m = \infty \\ XM_i^s + STB_i^m = \chi_i^m \left(\dfrac{PWM_i}{PWM_i^{RoW}}\right)^{\omega_i^m} & \text{if } \omega_i^m \neq \infty \end{cases} \tag{3.96}$$

There are potentially three goods market equilibrium conditions:

$$XD_i^s = XD_i^d$$

$$XE_i^s = XE_i^d$$

$$XM_i^s = XM_i^d$$

These equations are substituted away and the model implementation only carries a single variable for each of the equilibrium variables (XD, XE, and XM).

3.6 Factor market closure

3.6.1 Labor market equilibrium

The allocation of labor across activities uses a nested CET structure. First, an aggregate labor supply schedule for each skill type, LT^s, is specified as an upward-sloping supply curve—allowing for both extremes, a vertical curve with employment fixed, and a horizontal curve with wages tied to a price index. Equation (3.97) implements the national labor supply curve with the supply elasticity given by ϵ^l. The national or average wage is given by AW. Aggregate labor supply is then segmented into different markets, indexed by z, using a top level adjusted CET function. A standard model of labor market segmentation divides labor markets into two zones—rural and urban, or alternatively agriculture and non-agriculture. The adjusted CET function allows for mobility across labor markets in such a way that labor additivity is preserved.[25] The specification allows for perfect mobility in which case the average wage in each zone is equated up to a scalar, i.e. wages move in unison. A second level adjusted CET is used to allocate labor across activities within each zone—again allowing for perfect mobility.

Equation (3.98) determines aggregate labor supply for each zone. labor supply, LZ^s, where the parameter ω^l determines the degree of mobility across labor markets. In the case of perfect mobility, the wage in each zone moves in unison with the average wage index (not the average wage as in the standard CET). Equation (3.99) determines the national wage index, AW^n, that is used for the allocation of labor across zones. In the standard CET, the wage index and the aggregate wage, AW, are identical. To allow for the case of perfect mobility, the wage index equation is replaced with the additivity condition that holds for both perfect and partial mobility. Equation (3.100) determines the aggregate, or average wage rate, using the standard accounting identities.

$$\begin{cases} LT_l^s = \chi_l^{ls} \left(\dfrac{AW_l}{PNUM}\right)^{\epsilon_l^l} & \text{if } \epsilon_l^l \neq \infty \\ AW_l = PNUM & \text{if } \epsilon_l^l = \infty \end{cases} \tag{3.97}$$

$$\begin{cases} LZ_{z,l}^s = \gamma_{z,l}^{lz} \left(\dfrac{ZW_{z,l}}{\phi_{z,l}^{lz} AW_l^n}\right)^{\omega_l^l} LT_l^s & \text{if } \omega_l^l \neq \infty \\ ZW_{z,l} = \phi_{z,l}^{lz} AW_l^n & \text{if } \omega_l^l = \infty \end{cases} \tag{3.98}$$

25 Unlike the standard CET that does not preserve additivity. See the annex for more details on the adjusted CET.

$$AW_l^n = \left[\sum_z \gamma_{z,l}^{lz} \left(\frac{ZW_{z,l}}{\phi_{z,l}^{lz}} \right)^{\omega_l^l} \right]^{1/\omega_l^l} \iff LT_l^s = \sum_z LZ_{z,l}^s \tag{3.99}$$

$$AW_l LT_l^s = \sum_z ZW_{z,l} LZ_{z,l}^s \tag{3.100}$$

The second level adjusted CET allocates labor across activities within each zone. Equation (3.101) determines sectoral supply of labor to each activity, L^s. In the case of perfect mobility, the sectoral wage moves in unison with the wage index for the zone (not the average wage of the zone as in the standard CET). Equation (3.102) determines the aggregate wage index, ZW^n within each zone. It is replaced with the equivalent expression of volume additivity that holds for both partial and perfect mobility. Equation (3.103) determines the aggregate wage in each zone, ZW.

$$\begin{cases} L_{a,l}^s = \gamma_{a,l}^l \left(\dfrac{W_{a,l}}{\phi_{a,l}^l ZW_{z,l}^n} \right)^{\omega_{z,l}^{lz}} LZ_{z,l}^s & \text{if } \omega_l^{lz} \neq \infty \text{ and } a \in z \\ W_{a,l} = \phi_{a,l}^l ZW_{z,l}^n & \text{if } \omega_{z,l}^{lz} = \infty \text{ and } a \in z \end{cases} \tag{3.101}$$

$$ZW_{z,l}^n = \left[\sum_{a \in z} \gamma_{a,l}^l \left(\frac{W_{a,l}}{\phi_{a,l}^l} \right)^{\omega_{z,l}^{lz}} \right]^{1/\omega_{z,l}^{lz}} \iff LZ_{z,l}^s = \sum_{a \in z} L_{a,l}^s \tag{3.102}$$

$$ZW_{z,l} LZ_{z,l}^s = \sum_{a \in z} W_{a,l} L_{a,l}^s \tag{3.103}$$

Equation (3.104) is the equilibrium condition on labor markets where sectoral demand for labor equals sectoral labor supply. Equation (3.105) converts the market price of labor to the user-cost of labor by sector and skill.

$$L_{a,l}^s = L_{a,l}^d \tag{3.104}$$

$$W_{a,l}^p = \left(1 + \tau_{a,l}^l\right) W_{a,l} \tag{3.105}$$

3.6.2 Land market equilibrium

The model has a single national market for land. Aggregate land supply, $TLand$, is specified as an upward-sloping supply curve—allowing for both extremes, a vertical curve with total land fixed, and a horizontal curve with the return to land tied to a price index. Equation (3.106) implements the national land supply curve with the supply elasticity given by ϵ^t. The national or average price of land is given by $PTLand$. National supply is allocated to each sector under the assumption of perfect or sluggish mobility. In the case of perfect mobility, the return to land is uniform across sectors and moves in sync with the national price of land. With sluggish mobility, the return to land in each sector is tied to some extent to sectoral conditions and becomes sector specific. Equation (3.107) determines sectoral land supply, $Land^s$, where the parameter ω^t determines the degree of mobility. Equation (3.108) then defines the average national return to land. Equation (3.109) is the equilibrium condition on the land markets where sectoral demand for land equals sectoral land supply. Equation (3.110) converts the equilibrium price to the end-user price (i.e. producer cost) of land by sector.

$$\begin{cases} TLand = \chi^{tl} \left(\dfrac{PTLand}{PNUM} \right)^{\omega^{tl}} & \text{if } \omega^{tl} \neq \infty \\ PTLand = PNUM & \text{if } \omega^{tl} = \infty \end{cases} \tag{3.106}$$

$$\begin{cases} Land_a^s = \gamma_a^t \left(\dfrac{PLand_a}{PTLand} \right)^{\omega^t} TLand & \text{if } \omega^t \neq \infty \\ PLand_a = PTLand & \text{if } \omega^t = \infty \end{cases} \tag{3.107}$$

$$PTLand\, TLand = \sum_a PLand_a Land_a^s \tag{3.108}$$

$$Land_a^s = Land_a^d \tag{3.109}$$

$$PLand_a^p = \left(1 + \tau_a^t\right) PLand_a \tag{3.110}$$

3.6.3 Capital market equilibrium in comparative statics

There are two different implementations of capital market equilibrium in the model. The first is used for the comparative static version of the model and is virtually identical to the specification of the labor market. There is a national capital supply curve—and aggregate capital is allocated to sectors using a CET specification that is intended to capture the degree of inter-sectoral capital mobility. The second specification is intended for the dynamic version of the model. It includes a vintage specification for capital with a putty/semi-putty assumption.

Equations (3.111) through (3.114) reflect the capital market equations in the comparative static version of the model and replicate the labor equations above. The two key elasticities are the aggregate supply elasticity, ϵ^k, and the CET transformation elasticity, ω^k, that proxies the degree of intersectoral capital mobility. The equations are might by indexed by v, the index for vintages. In the comparative static version of the model there is a single vintage. Also note that equation (3.114) is substituted out of the model.

$$\begin{cases} TKAP^s = \chi^{Ks}\left(\dfrac{TR}{PNUM}\right)^{\epsilon^k} & \text{if } \epsilon^k \neq \infty \\ TR = PNUM & \text{if } \epsilon^k = \infty \end{cases} \tag{3.111}$$

$$\begin{cases} K^s_{a,v} = \gamma^k_a\left(\dfrac{PK_{a,v}}{TR}\right)^{\omega^k} TKAP^s & \text{if } \omega^k \neq \infty \\ PK_{a,v} = TR & \text{if } \omega^k = \infty \end{cases} \tag{3.112}$$

$$TR.TKAP^s = \sum_a PK_{a,v}K^s_{a,v} \tag{3.113}$$

$$K^s_{a,v} = K^d_{a,v} \tag{3.114}$$

3.6.4 Capital markets with the vintage capital specification

This section describes sectoral capital allocation under the assumption of multiple vintage capital and is used in the dynamic version of the model. Capital market equilibrium under the vintage capital framework assumes the following:

- *New* capital is perfectly mobile and its allocation across sectors insures a uniform rate of return.
- *Old* capital in expanding sectors is equated to new capital, i.e. the rate of return on *Old* capital in expanding sectors is the same as the economy-wide rate of return on new capital.
- Declining sectors release *Old* capital. The released *Old* capital is added to the stock of *New* capital. The assumption here is that declining sectors will first release the most mobile types of capital, and this capital, being mobile, is comparable to *New* capital (e.g. transportation equipment).
- The rate of return on capital in declining sectors is determined by sector-specific supply and demand conditions.

The result of these assumptions is that if there are no sectors with declining economic activity, there is a single economy-wide rate of return. In the case of declining sectors, there will be an additional sector-specific rate of return on *Old* capital for each sector in decline.

To determine whether a sector is in decline or not, one assesses total sectoral demand (which of course, in equilibrium equals output). Given the capital-output ratio, it is possible to calculate whether the initially installed capital is able to produce the given demand. In a declining sector, the installed capital will exceed the capital necessary to produce existing demand. These sectors will therefore release capital on the secondary capital market in order to match their effective (capital) demand with supply. The supply schedule for released capital is a constant elasticity of supply function where the main argument is the change in the relative return between *Old* and *New* capital. Supply of capital to the declining sector is given by the following formula:

$$K^s_{a,Old} = K^0_a\left(\frac{R_{a,Old}}{R_{a,New}}\right)^{\eta^k_a}$$

where K^s_{Old} is capital supply in the declining sector, K^0 is the initial installed (and depreciated) capital in the sector at the beginning of the period, and η^k is the dis-investment elasticity. (Note that in the model, the variable R is represented by PK.) In other words, as the rate of return on *Old* capital increases towards (decreases from) the rate of return on *New* capital, capital supply in the declining sector will increase (decrease). Released capital is the difference between K^0 and K^s_{Old}. It is added to the stock of *New* capital. In equilibrium, the *Old* supply of capital must equal the sectoral demand for capital:

$$K^s_{a,Old} = K^d_{a,Old}$$

Inserting this into the equation above and defining the following variable

$$RR_a = \frac{R_{a,Old}}{R_{a,New}}$$

yields the following equilibrium condition:

$$K^s_{a,Old} = K^0_a RR_a^{\eta^k_a}$$

The supply curve is kinked, i.e. the relative rate of return is bounded above by 1. If demand for capital exceeds installed capital, the sector will demand *New* capital and the rate of return on *Old* capital is equal to the rate of return on *New* capital, i.e. the relative rate of return is 1. The kinked supply curve has been transformed into a mixed complementarity (MCP) relation. The following inequality is inserted in the model:

$$K^s_{a,Old} = K^0_a RR_a^{\eta^k_a} \leq K_a^{d,Not} = \chi^v_a XP_a$$

The right-hand side determines the *notional* demand for capital in sector a, i.e. it assesses aggregate output (equal to demand) and multiplies this by the capital output ratio for *Old* capital. This is then the derived demand for *Old* capital. If the installed capital is insufficient to meet demand for *Old* capital, the sector will demand *New* capital, and the inequality obtains with the relative rates of return capped at 1. If the derived demand for *Old* capital is less than installed capital, the sector will release capital according to the supply schedule. In this case the inequality transforms into an equality, and the relative rate of return is less than 1.

Equation (3.115) determines the capital output ratio, χ^v for *Old* capital. Equation (3.116) specifies the supply schedule of *Old* capital. In effect, this equation determines the variable RR, the relative rate of return between *Old* and *New* capital.

$$\chi^v_{a,v} = \frac{K^d_{a,v}}{XPv_{a,v}} \tag{3.115}$$

$$K^0_a RR_a^{\eta^k_a} \leq \chi^v_{a,Old} XP_a \perp RR_a \leq 1 \tag{3.116}$$

There is a single economy-wide rate of return on *New* capital. The equilibrium rate of return on *New* capital is determined by setting aggregate supply equal to aggregate demand. Aggregate demand for *New* capital is given by:

$$\sum_{\{a|Expanding\}} \sum_v K^d_{a,v}$$

where the set *Expanding* includes all sectors in expansion. Since *Old* capital in expanding sectors is equated with *New* capital, the appropriate sum is over all vintages. The aggregate capital stock of *New* capital is equal to the total capital stock, less capital supply in declining sectors:

$$K^s - \sum_{\{a|Decling\}} K^s_{a,Old}$$

where the set *Declining* covers only those sectors in decline. However, at equilibrium, capital supply in declining sectors must equal capital demand for *Old* capital, and capital demand for *New* capital in these sectors is equal to zero. Hence, the supply of *Old* capital in declining sectors can be shifted to the demand side of the equilibrium condition for *New* capital, and this simplification yields equation (3.117) which determines the economy-wide rate of return on *New* capital, *TR*. Equation (3.118) determines the vintage and sector specific rates of return. For *New* capital, RR is 1 and thus the rate of return on *New* capital is always equal to the economy-wide rate of return. For *Old* capital, if the sector is in decline, RR is less than 1 and the rate of return on *Old* capital will be less than the economy-wide rate of return.

$$TKAP^s = \sum_a \sum_v K^d_{a,v} \tag{3.117}$$

$$PK_{a,v} = TR.RR_a \tag{3.118}$$

3.6.5 Allocation of Output across Vintages

This section describes how output is allocated across vintages. Aggregate sectoral output, XP, is equated to aggregate sectoral demand and is derived from XS, which itself is derived from a CET aggregation of XD and XE. Given the beginning of period installed capital, it is possible to assess the level of potential output produced using the installed capital. If this level of output is greater than the aggregate output (demand) level, the sector appears to be in decline, installed capital will be released, *Old* output will be equated with aggregate output (demand), and *New* output is zero. Equation (3.119) determines output that can be derived from installed, or *Old* capital. *Old* output is equated to the sectoral supply of *Old* capital, divided by the capital/output ratio. Equation (3.120) equates aggregate output, XP, to the sum of output across all vintages, thus in essence this equation determines output produced with *New* capital by residual since equation (3.119) determines what can be produced with *Old* capital. Equation (3.121) converts the market price of capital to the end-user cost of capital.

$$XPv_{a,Old} = \frac{K_a^0}{\chi_{a,Old}^v} RR_a^{\eta_a^k} \tag{3.119}$$

$$XP_a = \sum_v XPv_{a,v} \tag{3.120}$$

$$PK_{a,v}^p = \left(1 + \tau_{a,v}^k\right) PK_{a,v} \tag{3.121}$$

3.6.6 Sector-specific capital

For the moment, sector-specific capital is assumed fixed and the equilibrium condition is implicit in the formulation, i.e. the demand equation determines the equilibrium return. There is a wedge between the equilibrium price and the return to agents given by the direct tax κ^{kf}. Equation 3.122 determines the net return to private agents, with the residual accruing to public fiscal authorities.

$$RKF_a^n = \left(1 - \kappa_a^{kf}\right) RKF_a \tag{3.122}$$

3.6.7 Natural resource market equilibrium

Natural resources are sector specific. Their supply is given by an iso-elastic supply curve, equation (3.123), where ω^{nr} is the supply elasticity. The specification allows for a horizontal supply curve where the price is equated to a domestic price index. The shifter χ^{nr} can be used to calibrate the supply equation to an exogenous profile for the price of the natural resource (or its associated output). Equation (3.124) equates supply of the natural resource to its demand.[26]

$$\begin{cases} XNR_a^s = \chi_a^{nr} \gamma_a^{nr} \left(\dfrac{PNR_a}{PNUM}\right)^{\omega_a^{nr}} & \text{if } \omega_a^{nr} \neq \infty \\ PNR_a = PNUM & \text{if } \omega_a^{nr} = \infty \end{cases} \tag{3.123}$$

$$XNR_a^s = XNR_a \tag{3.124}$$

3.7 Macro identities

The following block of equations provides the main macroeconomic identities starting with the components of GDP at market prices. Equations (3.125) through (3.127) determine respectively nominal aggregate stock building, real aggregate stock building the stock building aggregate price index.

$$TSTB = \sum_i \left[\chi_i^{PS} PS_i STB_i^d + PM_i^d STB_i^m\right] \tag{3.125}$$

$$RTSTB = \sum_i \left[\chi_i^{PS} PS_{i,0} STB_i^d + PM_{i,0}^d STB_i^m\right] \tag{3.126}$$

$$PTSTB = TSTB/RTSTB \tag{3.127}$$

[26] This equation is substituted out of the model, which only incorporates the variable XNR.

Equations (3.128) through (3.130) determine respectively the nominal and real values for aggregate exports and the aggregate export price index. Aggregate exports are evaluated at border, or FOB, prices in domestic currency units.

$$TEXP = ER \sum_i \left[\chi_i^{PWE} PWE_i XE_i \right] \tag{3.128}$$

$$RTEXP = ER_0 \sum_i \left[\chi_i^{PWE} PWE_{i,0} XE_i \right] \tag{3.129}$$

$$PEXP = TEXP / RTEXP \tag{3.130}$$

Equations (3.131) through (3.133) determine respectively the nominal and real values for aggregate imports and the aggregate import price index. Aggregate imports are evaluated at border, or CIF, prices in domestic currency units.

$$TIMP = ER \sum_i \left[PWM_i XM_i \right] \tag{3.131}$$

$$RTIMP = ER_0 \sum_i \left[PWM_{i,0} XM_i \right] \tag{3.132}$$

$$PIMP = TIMP / RTIMP \tag{3.133}$$

Equation (3.134) defines nominal GDP at market price, where the index *fd* in the sum covers the standard final demand accounts (households (h), government current expenditures ('*Gov*') and investment expenditures ('*Inv*')). Equation (3.135) defines real GDP at market price, with the GDP at market price deflator given in equation (3.136). Per capita real GDP is defined in equation (3.137), where $TPop$ is aggregate population. Equation (3.138) is relevant in dynamic simulations. In the baseline simulation, the growth in real per capita GDP, g^y, may be exogenous (i.e. targeted) and another variable would be endogenous to achieve the growth target—typically some factor productivity parameter (discussed further below). In policy or alternative scenarios, the growth in real per capita GDP would be endogenous.

$$GDPMP = \sum_{fd} YF_{fd} + TSTB + TEXP - TIMP \tag{3.134}$$

$$RGDPMP = \sum_{fd} XF_{fd} + RTSTB + RTEXP - RTIMP \tag{3.135}$$

$$PGDPMP = GDPMP / RGDPMP \tag{3.136}$$

$$RGDPMPPC = RGDPMP / TPop \tag{3.137}$$

$$g_t^y = \left(\frac{RGDPMPPC_t}{RGDPMPPC_{t-n}} \right)^{1/n} - 1 \tag{3.138}$$

Equation (3.139) defines nominal GDP at factor cost—it is simply the sum of factor remuneration across all factors and activities of production. Equation (3.140) defines real GDP at factor cost. It is a linearization of the true GDP function and is the weighted sum of the factors of production in efficiency units where the weights are given by the relevant base year prices. The GDP at factor cost deflator is defined in equation (3.141).

$$\begin{aligned} GDPFC &= \sum_a \sum_l W_{a,l} L_{a,l}^d + \sum_a \sum_v PK_{a,v} K_{a,v}^d \\ &+ \sum_a PLand_a Land_a^d + \sum_a PNR_a XNR_a \end{aligned} \tag{3.139}$$

$$\begin{aligned} RGDPFC &= \sum_a \sum_l W_{a,l,0} \lambda_{a,l}^l L_{a,l}^d + \sum_a \sum_v PK_{a,v,0} \lambda_{a,v}^k K_{a,v}^d \\ &+ \sum_a PLand_{a,0} \lambda_a^t Land_a^d + \sum_a PNR_{a,0} \lambda_a^{nr} XNR_a \end{aligned} \tag{3.140}$$

$$PGDPFC = GDPFC / RGDPFC \tag{3.141}$$

The final macro identities provide more flexibility with closure rules. Key macroeconomic shares, real and nominal government and investment expenditures as a share of GDP, are described in Equations (3.142) and (3.143) respectively.

$$\varphi_f^r = 100\frac{XF_f}{RGDPMP} \tag{3.142}$$

$$\varphi_f^n = 100\frac{YF_f}{GDPMP} \tag{3.143}$$

Equation (3.144) defines the Fisher ideal price index for factor prices. It is based on the following formula that defines the value of factor remuneration priced at time *tp* and using quantities from period *tq*:

$$\begin{aligned} VFACT_{tp,tq} &= \sum_a\sum_l W_{a,l,tp}L^d_{a,l,tq} + \sum_a\sum_v PK_{a,v,tp}K^d_{a,v,tq} \\ &+ \sum_a PLand_{a,tp}Land^d_{a,tq} + \sum_a PNR_{a,tp}XNR_{a,tq} \end{aligned}$$

The factor price index is a candidate to swap with foreign savings in a closure where the real exchange rate is fixed. In this closure, $PFACT$ is exogenous, i.e. the average domestic factor price is fixed relative to external prices (assuming the exchange rate is the model numéraire) and the capital account adjusts to *ex ante* changes to the real exchange rate. For example, if the world price of a key import, such as oil, increases, under the normal closure, the real exchange rate adjusts (typically depreciates) to increase exports to pay for the higher cost of imports. In the alternative closure, the capital account adjusts, for example through more borrowing, to pay for the higher cost of imports.

$$PFACT_t = PFACT_{t-1}\sqrt{\frac{VFACT_{t,t-1}}{VFACT_{t-1,t-1}}\cdot\frac{VFACT_{t,t}}{VFACT_{t-1,t}}} \tag{3.144}$$

Equation (3.145) defines the domestic price deflator that is used in various equations. Other aggregate price indices could be chosen as an alternative.

$$PNUM = PGDPMP \tag{3.145}$$

Equation (3.146) defines equivalent variation. It represents the level of income, EV, necessary to achieve contemporaneous utility level, u, at base prices, PKF_0. It is a measure of welfare for households. Total economy-wide welfare needs to take into account public and investment expenditures.[27]

$$\sum_k \alpha^{kf}_{h,k} u_h^{e_{h,k}b_{h,k}}\left(\frac{PKF_{h,k,0}}{EV_h Pop_h}\right)^{b_{h,k}} \equiv 1 \tag{3.146}$$

Due to Walras' Law, one equation in the model can be dropped. The current implementation drops the balance of payments equation (3.147). It is incorporated in the model but the variable Walras should be zero if the simulation has succeeded and this is part of the model consistency check. Note that the balance of payments equation is evaluated in foreign currency units and thus outbound transfers are divided by the exchange rate.

$$\begin{aligned} Walras &= \underbrace{\sum_i \chi_i^{PWE} PWE_i XE_i}_{\text{Exports}} + \underbrace{S^f}_{\text{Net investment flows}} + \underbrace{\sum_{inst} Transfers_{inst,row}}_{\text{Inbound transfers}} \\ &- \underbrace{\sum_i PWM_i\left(XM_i + STB_i^m\right)}_{\text{Imports}} - \underbrace{\sum_{inst} Transfers_{row,inst}/ER}_{\text{Outbound transfers}} \end{aligned} \tag{3.147}$$

The default closure rules of the model are as follows:

- Household savings are endogenous.[28]
- Government revenues are endogenous and aggregate real government expenditures are fixed. The government balance is fixed, in part to avoid problems of financing sustainability. The government balance is achieved with a uniform shift in the household direct tax schedule—currently implemented using the additive shifter (though with a single household model this has no implications on distribution). This implies that new revenues, for example generated by a carbon tax, would lower direct taxes paid by households.

[27] NEW: Added to the model on 31-May-2018.
[28] Though given the current model implementation the average propensity to save is fixed.

- Investment is savings driven. Household and government savings were discussed above. Foreign savings, in the default closure are fixed. Thus investment is largely influenced through household savings.[29]
- The current account, the mirror entry of the capital account, is exogenous. *Ex ante* changes to trade, for example a rise in the world price of imported oil, is met through *ex post* changes in the real exchange rate.

3.8 Emissions module

The model is setup so as to include any number of emissions, indexed by em, generated by input use (both intermediate and factor demand—for example livestock or land), and by output (for example the case of cement, or methane emissions from landfills). The current version only incorporates CO_2 emissions—mostly from the combustion of fossil fuels, though with some quantity generated by cement producing emissions.

Equation (3.148) determines the level of emissions from the (Armington) consumption of good i, by Armington agent aa (either intermediate or final demand). The basic coefficient is ρ that is the initial level of emissions per unit of consumption (for example, tons of CO_2 per ton of oil equivalent). The χ parameters allow for (exogenous) changes in the emissions coefficients that could be brought about by autonomous improvements in the level of emissions per unit of use. Equation (3.149) determines the level of emissions per unit of factor use. In livestock this could pertain to the size of the herds (for example for methane emissions), and in agriculture, it could be linked to land use. In the current version of the model, only capital use is allowed to create emissions and at the moment, all emissions are zero. Equation (3.150) determines the level of emissions generated by overall output. This can be used to assess the level of emissions from certain sectors—such as CO_2 emissions in cement (around 50 percent of which are not linked to the use of fuels), or methane emissions from landfills. Equation (3.151) determines the total level of emissions, for emission em, summing over all sources—intermediate and final consumption, and factor- and production-based emissions. It allows for an exogenous level of autonomous emissions—that could, for example, come from an external model. A subset of emissions are linked with greenhouse gases (ghg) such as CO_2, methane, nitrous oxides and the fluoridated gases. Equation (3.152) calculates the total of greenhouse gas emissions where the individual gases are weighted by the so-called global warming potential (GWP).[30]

$$EMI_{em,i,aa} = \chi^{emi}_{em,i,aa}\rho^{emi}_{em,i,aa}XA_{i,aa} \tag{3.148}$$

$$EMI_{em,cap,a} = \chi^{emi}_{em,cap,a}\rho^{emi}_{em,cap,a}\sum_v K^d_{a,v} \tag{3.149}$$

$$EMI^{XP}_{em,cap,a} = \chi^{emiXP}_{em,a}\rho^{emiXP}_{em,a}XP_a \tag{3.150}$$

$$EMITot_{em} = \sum_i\sum_{aa} EMI_{em,i,aa} + \sum_{fp}\sum_a EMI_{em,fp,a} + \sum_a EMI^{XP}_{em,a} + EMIOth_{em} \tag{3.151}$$

$$EMI^{GHG} = \sum_{ghg} GWP_{ghg}EMITot_{ghg} \tag{3.152}$$

There are several possibilities for imposing limits on emissions. The simplest is to set a level for the emissions tax, τ^{em}. At the moment, τ^{em} is only linked to the direct consumption of (Armington) goods, i.e. it is only impacted through Equation (3.148), with no effect on the use of factors or production level. [We would need to develop marginal abatement curves to directly address these latter two. In the case of factors, the only potential abatement is substitution to other factors. In the case of production, the only abatement possibility is a reduction in output.] An alternative is to cap emissions, for example $EMITot$, and allow the model to calculate the relevant tax (e.g. carbon tax), consistent with the emissions constraint. With the standard closure, the marginal tax rates across households are shifted by a uniform amount as government expenditures and savings are exogenous.

3.9 Miscellaneous equations

Equations (3.153) and (3.154) are introduced to allow for some control of the electricity mix. Equation (3.153) defines the (volume) electricity shares in the production of electricity (where the index ely is mapped to the national electricity commodity). Equation (3.154) defines the aggregate share of renewable electricity (where the subset $gEly$

---

29 Alternative closures are conceivable, for example targeting investment (as a share of GDP) and allowing the household savings schedule adjust to achieve the target.

30 See IPCC (1996), page 22, and also http://unfccc.int/ghg_data/items/3825.php.

contains the renewable electricity technologies). The latter can be fixed to attain a given renewable mandate for electricity production. Only one instrument can be used, even if the renewable target encompasses more than one technology.

$$elyShr_a = \frac{X_{a,ely}}{\sum\limits_{a'} X_{a',ely}} \tag{3.153}$$

$$gElyShr = \sum_{a \in gEly} elyShr_a \tag{3.154}$$

3.10 Model Dynamics

Model dynamics are driven by three factors—similar to most neo-classical growth models. Population and labor force growth rates are exogenous. The labor force growth rate is typically equated to the growth rate of the working age population, i.e. the population aged between 15 and 64.

The second factor is capital accumulation. The aggregate capital stock in any given year, *KStock*, is equated to the previous year capital stock, less depreciation at a rate of δ, plus the previous period's volume of investment, XF_{Inv}:

$$KStock_t = (1-\delta) KStock_{t-1} + XF_{Inv,t-1}$$

The latter is influenced by the national savings rate plus foreign savings and, as well, the unit cost of investment. A modified version of the capital accumulation function is implemented to allow for multi-period steps in the dynamic simulations (see Appendix C). Since investment is not calculated between periods, an assumption is made about the inter-period growth rate in investment and this is used to determine the contemporaneous capital stock. Equations (3.155) and (3.156) implement the multi-period capital stock motion equation. Setting n to 1 shows that the combination of the two equations is equivalent to the equation above. The aggregate capital stock variable takes two forms. The first, *KStock*, is the aggregate capital stock evaluated at base year prices. The second is the 'normalized' aggregate capital stock, $TKAP^s$, see equation (3.117). The normalized capital stock is equal to the aggregate base year capital remuneration, i.e. the user cost of capital across sectors. It is normalized because its price is set to 1 in the base year. The ratio of the normalized capital stock to the actual capital stock provides a measure of the gross rate of return to capital. It is assumed that both measures of the capital stock grow at the same rate and hence equation (3.157) that equalizes the ratio of the two measures.[31]

$$IGFact_t = \left[\left(\frac{XF_{Inv,t}}{XF_{Inv,t-n}} \right)^{1/n} - 1 + \delta_t \right]^{-1} \tag{3.155}$$

$$KStock_t = \left[KStock_{t-n} - IGFact_t XF_{Inv,t-n} \right] (1-\delta_t)^n + IGFact_t XF_{Inv,t} \tag{3.156}$$

$$TKAP_t^s = \frac{TKAP_0^s}{KStock_0} KStock_t \tag{3.157}$$

The third factor is productivity. There are a number of productivity factors peppered throughout the model. The key productivity factor is λ^l that corresponds to converting labor in volume terms to labor in efficiency units. It is typically initialized at 1 in the base year. The following assumptions are made regarding productivity:

- Sectors are segmented into three groups—those for which productivity is fully exogenous, those for which an economy-wide productivity is calibrated, and others that are based on the calibrated economy-wide productivity, but with an additional wedge. For example labor productivity in services might be calibrated to achieve a growth target and productivity in manufacturing might be some function of services productivity (say 2 percentage points higher).
- Typically productivity in agriculture is exogenous and factor neutral. The λ^n and λ^v parameters are set to grow at some exogenous and uniform rate.
- In the other sectors, productivity is labor augmenting only.

31 It is important to use the actual capital stock in the capital accumulation function since the level of investment must correspond to the actual capital stock, not the normalized level.

- There is a wedge between productivity in the two remaining segmented sectors (e.g. manufacturing and services), represented by the factors α and β in equation (3.158). For some set of sector(s) and skill level(s), α is 0 and β is 1. The economy-wide labor productivity shifter, γ^l, is calibrated to achieve a given growth target in the business-as-usual scenario. In the other sector(s) and for other skill level(s) the labor productivity will be some linear combination of γ^l. For example, if α is 2 percent (0.02) and β is 1, then the wedge would be 2 percent.

$$\lambda^l_{a,l,t} = \lambda^l_{a,l,t-n}\left(1+\alpha^l_{a,l}+\beta^l_{a,l}\gamma^l_t\right)^n \tag{3.158}$$

The other key growth/efficiency parameter in the model is energy efficiency as captured by the λ^e parameter in production and the λ^{eh} and λ^{ef} parameters in the final demand for energy. These are defined exogenously—for example 1 percent per annum.

Chapter 4

Model Implementation

4.1 Model files

Table 4.1 lists the core files used for model implementation:

Table 4.1: Code model files

Name	**Description**
model.gms	File containing model definition
cal.gms	File to initialize variables and calibrate model parameters
iterloop.gms	File implemented before each solve statement
postsim.gms	File containing post-simulation statements

The central file is called **model.gms** and contains variable, parameter and equation declarations followed by equation and model definitions. The listing of the model equations follows the same order as in this document and should allow for a relatively transparent comparison of the model write-up and its implementation in GAMS. The **cal.gms** file contains variable initialization and calibration of model parameters. Like most CGE models, many model parameters are calibrated such that the model implementation is able to re-produce the base dataset. The input to the calibration procedure is the base year SAM, potentially some satellite accounts such as energy balances, population, etc. and a set of key parameters (mostly elasticities). More formally the model can be written as:

$$F(y, x, \theta, \Omega) = 0$$

where y is a multi-dimensional set of endogenous variables, x is a multi-dimensional set of exogenous variables, and the parameter set is divided into two: θ contains calibrated parameters and Ω contains key (or user-specified) parameters. A typical simulation is then solving F for y with x, θ and Ω fixed—and x deviates from its base value. In the calibration phase, y, x and Ω are known and the function F is used to solve for θ:

$$F(y_0, x_0, \theta, \Omega) = 0$$

In practice, calibration can be done block by block without the need to invert F formally.

The file **iterloop.gms** contains code that is called before each solution period. It will fix zero activities, fix lagged variables, update exogenous dynamic variables and implement closure rules.

Any simulation—be it comparative static or dynamic—requires these three files. A specific simulation is attached to a given simulation file. In addition to the three core model files, a simulation file will read in the input data files—the base year SAM and satellite accounts, user-specified parameters, and, in the case of dynamic scenarios, a file containing dynamic information, e.g. reference population and GDP trends.

4.2 Model simulation

The model can be run as either a comparative static model or as a recursive dynamic model. The former is often used to validate model initialization, calibration and consistency—particularly when testing new model features. The latter is more often used for forward looking policy analysis or for dynamic sensitivity analysis. Whether doing comparative statics or dynamic simulations, a simulation file has to be compiled by the user that brings together the different components of the model:

1. Read the base data
2. Aggregate the base data and read the key model elasticities
3. Define the *time* dimension of the model
4. Read in the model code (**model.gms**)
5. Read in the dynamic assumptions
6. Initialize and calibrate the model (**cal.gms**)
7. Loop over all *time* periods:
   - Initialize exogenous assumptions (**iterloop.gms**)
   - Define time-specific shocks
   - Solve the model
   - Save intermediate results
8. Post-simulation analysis

There are two generic files that provide the structure of a standard simulation—one for comparative statics (**comp.gms**) and one for a dynamic baseline (**bau.gms**). The differences will be highlighted below. There are two additional dynamic files. One is called **dynNoShk.gms**. The purpose of this file is to re-run the baseline dynamic file (**bau.gms**) with no shocks. The difference between this simulation and the baseline simulation is that some dynamic variables are calibrated in the baseline scenario (e.g. government expenditures and labor productivity). The **dynNoShk.gms** simulation file is intended to check the consistency of the dynamic scenario in the absence of any shock. A third dynamic file is called **dynShk.gms**. For the moment it is simply a test file that simulates a reduction in import tariffs.

4.2.1 Comparative static simulation file

The GAMS code below shows the structure of the comparative static simulation file. The comments below highlight the key elements that the user might typically change in a comparative static simulation.

1. The first lines read in the base data. This will be described further below. The first data set contains the full base year SAM. The second contains potential aggregation mappings of the full SAM, the definition of subsets needed for the model, and the user-specified key input parameters. The first file is rarely modified. The second may be modified by the user.[1]
2. The time dimension of the comparative static simulation is simply the number of simulations that will be run within the confines of this simulation file. In the standard **comp.gms** file, the time dimension has three items: *base*, *check*, and *shock*. The first item is typically not actually simulated and simply contains the initialized base year data for the model. The second, *check*, is actually used to test the consistency of data initialization and calibration. There are two checks normally performed. The first check requires analyzing the listing file produced by GAMS. Before running a simulation GAMS will list all of the equations and compare the left-hand side (LHS) of the model with the evaluated right-hand side.[2] If the data and parameters have been appropriately initialized and calibrated, the equations should evaluate to zero, or a very small number that represents the numerical accuracy of the input SAM (appropriately scaled). The second check is to look at the output results. The output of the *check* simulation should line up with the output of the non-simulated base

[1] This document, among other things, explains how to use MANAGE to simulate a single country extraction of the GTAP database. Assuming the data for a single country has been extracted, the user needs to define the `REGION` macro by using the relevant three-letter ISO code for the region.

[2] To make sure the evaluated equation listing appears in the list file, the `limrow` options must be set to a positive value. By default, if an equation is indexed, GAMS will display up to three equations. The option can be set by the following command: `options limrow=x ;`, where by default `x` is 3.

results. The third simulation, *shock*, typically tests the price homogeneity of the model.[3] This is often useful for debugging purposes in capturing model mis-specification. The results of the homogeneity test should be that all prices and values should increase by the same percentage change of the numéraire and that all volumes should be invariant.

3. The next section initializes the output files. Typically, the user may want to change the name of the output files. The standard output format is a CSV file that can be read directly into Excel, or more commonly and convenient, read in to an Excel Pivot table.

4. The next section should normally not be modified. It sets up the vintage specification for capital. In the comparative static model, there is only *Old* capital.

5. Most of the remainder of the file should not be modified with the exception of the actual shocks. The only shock in the default comparative static file is the modification of the numéraire. This is specified by the line: `er.fx(ts) = 1.2*er.l(ts) ;`.

The best way to run additional comparative static simulations is to copy the file **comp.gms** to a new file name and modify the *time*, output file names and the definition of the shock.

Listing 4.1: **Comparative static simulation file**

```
$setGlobal REGION "ISO"

$offlisting
$onempty

$include '%REGION%SAMV9.dat'
$include '%REGION%Base.dat'

$offempty
$onlisting

sets
   t         "Time"            / base, check, shock /
   t0(t)     "Base period"     / base /
   ts(t)     "Simulation period"
   tsim(t)   "This simulation period"
;

ts(t) = no ;
alias (t,tt) ;

parameter years(t) /
   base     1
   check    2
   shock    3
/ ;

parameter gap(t) ; gap(t) = 1 ;

file csv / %region%comp.csv / ;
put csv ;
put "Simulation,Variable,Sector,Qualifier,Year,Value" / ;
csv.pc=5 ;
csv.nd=9 ;

file samcsv / %region%compsam.csv / ;
scalar ifSam / 1 / ;
if(ifSam,
   put samcsv ;
   put "Simulation,rLab,cLab,Year,Value" / ;
   samcsv.pc=5 ;
   samcsv.nd=9 ;
) ;
```

3 This assumes that model is intended to be homogeneous in prices. There might be versions of the model, for example with a fixed price, in which case price homogeneity fails.

```
sets
   v       "Vintages"         / Old /
   v0(v)   "Initialization"   / Old /
   vc(v)   "Calibration"      / Old /
   vOld(v) "Old vintage"      / Old /
   Old(v)  / Old /
   New(v)  / Old /
;

set mapv(v, vint) /
   Old.Old
   Old.New
/ ;

scalar ifVint / 0 / ;

set diag /
   modStatus
   Walras
/ ;

parameters
   diagnostics(diag, t)
;

file screen / con / ;
put screen ;

*  Set ifComp to 1 for comparative static simulations

scalar ifComp / 1 / ;

*  Set ifCal to 1 for BaU scenarios

scalar ifCal / 0 / ;

*  Set ifCDE to 1 for using the CDE utility function

scalar ifCDE / 1 / ;

*  Closure rules

set cr /

*  Default -- government expenditures are fixed in real terms
*             government savings are fixed in real terms
*             household tax shifter is endogenous
*             investment is savings driven
*             public and foreign savings are fixed
*             household savings shifter is fixed
*             trade balance is fixed, and gdp deflator is endogenous

   default

/ ;

parameter closure(cr) ; closure('default') = yes ;

$include 'model.gms'

$include 'compscn.dat'

$include 'cal.gms'

loop(tt,

   if (ord(tt) gt 1,
```

```
      ts(tt) = yes ;

$include 'iterloop.gms'

      if (ts('shock'),

*        Homogeneity check

         er.fx(ts) = 1.2 ;

*        Labor productivty check
*        lambdal.fx(a,l,ts) = 1.1 ;

      ) ;

      options limrow=3, limcol=3 ;
*     options iterlim = 10 ;

      solve cgecore using mcp ;

      diagnostics("modStatus",ts) = cgecore.solvestat ;
      diagnostics("Walras",ts) = walras.l/inscale ;

      if (cgecore.solvestat eq 1,
         put screen ;
         put // "Solved for year ", years(tt):4:0 // ;
         tsim(ts) = yes ;
         loop(diag, put diag.tl:<10, diagnostics(diag,tt) / ; ) ;
         putclose screen ;
      else
         put screen ;
         put // "Failed to solve for year ", years(tt):4:0 // ;
         tsim(ts) = no ;
         loop(diag, put diag.tl:<10, diagnostics(diag,tt) / ; ) ;
         putclose screen ;
      ) ;

      ts(tt) = no ;

   ) ;

) ;

display diagnostics ;

set sim / comp / ;
loop(sim,

$include "postsim.gms" ;

) ;
```

4.2.2 Running dynamic simulations

Dynamic simulations require considerably more input from the user—notably a time framework for the simulations and key dynamic assumptions such as population growth. There are also two distinct types of dynamic simulations. There is a baseline simulation.[4] In the standard baseline simulation GDP growth is exogenous and given by the user. It might come from national planning authorities, a finance ministry or some other source. And other closure rules might be set that differ from the standard closure rules and/or trends for other exogenous variables are implemented—for example known policy changes, energy prices, foreign inflows and outflows, etc. The baseline simulation is used to calibrate an economy-wide productivity variable that ensures that the baseline results are consistent with the exogenous inputs including assumptions on GDP growth. In subsequent dynamic simulations, the calibrated

[4] Sometimes referred to as the reference simulation or the business-as-usual (BaU) simulation.

productivity variable is exogenous and GDP growth is endogenous—corresponding to changes in other variables—for example a policy shock. In the absence of any shocks, a dynamic scenario should be able to re-produce the baseline.

The dynamic simulation files are divided into two parts. There is a file that will be common across all dynamic simulations and thus will be kept separated and typically needs no change across the simulations. The other part of the dynamic simulation is what distinguishes the particular simulation. For example, there might be three separate dynamic simulations—the baseline, a no-shock simulation that should re-produce the baseline, and a dynamic shock simulation that deviates from the baseline. The description of the common part of a dynamic file is described first—the file is named **dynamDef.gms**

1. The first part of the common dynamic file describes the sets needed for the vintage part of the model.
2. The next section describes the full potential time horizon. This is linked to the SSP projections that were initially developed for GTAP V8 with a 2007 base year. It is still valid for GTAP V9 with a 2011 base year. The full time range thus spans the period 2007–2100.
3. The third section describes the SSP dimensions. The shared socio-economic pathways, or SSPs, have been developed by a number of research teams for climate change work related to the Intergovernmental Panel on Climate Change (IPCC). The dynamic simulations for the moment rely on the demographic and GDP quantifications of the SSPs. IIASA has been responsible for developing the demographic projections for the 5 SSPs. For our purposes these have been collapsed to three age cohorts (less than 15, 15 to 64 and 65 and above) and the gender and education dimensions are ignored. The IIASA population projections have been complemented with the demographic projections from the United Nations Population Division. There are two sets of UN projections—the 2010 and 2012 revision, for the medium variant. IIASA's SSP2 projection was intended to more or less align with the UN 2010 revision.

   Three research teams developed GDP projections—all of which were harmonized to the same IIASA population projections. The IIASA and OECD GDP projections were done at the individual country level. The third set of projections from PIK was done at a more aggregate regional level. For our purposes, only the IIASA and OECD GDP projections have been made available.

   There are five basic SSP scenarios that are summarized in Table 4.2.

   Table 4.2: SSP Scenarios

SSP	Description
SSP1	*Sustainability*—less resource intensity, equitable, good growth
SSP2	*BaU*—muddling through or dynamics as usual
SSP3	*Fragmentation*—with high inequality and poverty, low and dirty growth
SSP4	*Inequality*—highly unequal world but significant mitigation capacity
SSP5	*Conventional development*—relatively dirty but with high adaptation capacity

   One can think of SSP2 as a business-as-usual, SSP1 is a rosy scenario both in terms of sustainability and rapid and equitable growth, SSP3 is the polar opposite and SSP4 and SSP5 are variants. These scenarios are complete trends between 2007 and 2100 and therefore provide full flexibility in terms of the time dimension of the dynamic simulations. [We could add to these scenarios population/GDP scenarios prepared by national authorities or from other sources.]
4. The fourth section inputs the key data files of which there are three. Two are exactly the same as for the comparative static simulations, i.e. the base year SAM and the user defined aggregation mappings and key model elasticities. The third, **ISOBaUScn.dat**, contains the SSP information extracted for the relevant country. The file contains the full set of SSP information, only a subset of which will be needed for the dynamic simulation.
5. The remainder of this file is mostly self-explanatory. It is important to set the user flag `ifComp` to 0 for dynamic simulations. This largely affects how capital markets work. In comparative static simulations the capital market uses an upward sloping supply curve for aggregate capital and a CET function to allocate capital across sectors. In the dynamic setting, capital allocation depends on the vintage specification as described above.

   The last few lines of the file are only executed for non-baseline simulations, i.e. when the user flag `ifCal` is set to 0. For example, the baseline solution will be loaded into memory as a potential starting point.

Listing 4.2: **The common part of all dynamic simulations**

```
sets
   v       Vintages          / Old, New /
   v0(v)   Initialization    / Old /
   vc(v)   Calibration       / New /
   vOld(v) Old vintage       / Old /
   Old(v)  Old index         / Old /
   New(v)  New index         / New /

;

set tf Time range / 2007*2100 / ;

parameter
   years(tf)
;

years(tf) = 2006 + ord(tf) ;

set Scen Scenarios /
   ssp1      "Sustainability--less resource intensity, equitable, good growth"
   ssp2      "BaU--muddling through or dynamics as usual"
   ssp3      "Fragmentation--with high inequality and poverty, low and dirty growth"
   ssp4      "Inequality--highly unequal world but significant mitigation capacity"
   ssp5      "Conventional development--relatively dirty but with high adaptation capacity"
   UNMed     "UN Medium variant 2010 revision"
   UNMed2012 "UN Medium variant 2012 revision"
/ ;

set sspScen(Scen) SSP scenarios /
   ssp1      "Sustainability -- less resource intensity, equitable, good growth"
   ssp2      "BaU -- muddling through or dynamics as usual"
   ssp3      "Fragementation -- with high inequality and poverty, low and dirty growth"
   ssp4      "Inequality -- highly unequal world but significant mitigation capacity"
   ssp5      "Conventional development -- relatively dirty but with high adaptation capacity"
/ ;

set Mod GDP models /
   IIASA     "IIASA long term GDP projections"
   OECD      "OECD long term GDP projections"
/ ;

$offlisting
$onempty
$include '%REGION%SAMV9.dat'
$include '%REGION%Base.dat'
$include '%REGION%BaUScn.dat'
$offempty
$onlisting

set ts(t) Simulation period ; ts(t) = no ;
set tsim(t) ;
alias (t,tt) ;

set mapv(v, vint) /
   Old.Old
   New.New
/ ;

scalar ifVint / 1 / ;

set diag /
   modStatus
   Walras
/ ;

parameters
```

```
   diagnostics(diag, t)
;

$offsymxref offsymlist

file screen / con / ;
put screen ;

*  Set ifComp to 1 for comparative static simulations

scalar ifComp / 0 / ;


*  Set ifCDE to 1 for using the CDE utility function

scalar ifCDE / 1 / ;

*  Closure rules

set cr /

*  Default -- government expenditures are fixed in real terms
*             government savings are fixed in real terms
*             household tax shifter is endogenous
*             investment is savings driven
*             public and foreign savings are fixed
*             household savings shifter is fixed
*             trade balance is fixed, and gdp deflator is endogenous

   default

/ ;

parameter closure(cr) ; closure('default') = yes ;

$include 'model.gms'

$include 'cal.gms'

if(ifCal eq 0,

$if exist "%BAUNAME%.gdx" execute_loadpoint "%BAUNAME%.gdx" ;

   xfT(oa,t) = xf.l(oa,t) ;

) ;
```

The key component of the baseline simulation file is the **ifCal** flag it determines the type of dynamic simulation. When it is set to 1, the model assumes that GDP is exogenous and will calibrate an economy-wide productivity factor to insure that the GDP target is achieved. It is also assumed in the reference scenario that the share of government expenditures relative to GDP is constant. With *ifCal* set to 0, the productivity trend is read in from the GDX file that contains the results from the reference scenario. In addition, real government expenditures are also exogenous (in levels, not as a share of GDP), and likewise read in from the baseline GDX file.[5]

The main modules of the baseline simulation file are described next.

1. The first few lines initialize some macros that are used to name key files. The **REGION** macro is typically a 3-letter ISO card for the relevant country. The **SIMNAME** macro will provide a succinct descriptor of the simulation. The **BAUNAME** macro provides the name of the baseline scenario. The latter two macros could have the same name.
2. The next item sets the **ifCal** flag. It should only be set to 1 for the baseline scenario as described above.
3. After reading the common dynamic module, the baseline simulation loops over all time periods and solves the model once for each solution period. The statements in **iterloop.gms** provide a bridge between solution periods to update relevant variables and provide a starting point for the next simulation year.

[5] This assumption simplifies welfare analysis.

Listing 4.3: **The baseline simulation file**

```
$setGlobal REGION "ISO"
$setGlobal SIMNAME "BaU"
$setGlobal BAUNAME "BaU"

*  Set ifCal to 1 for BaU scenarios

scalar ifCal / 1 / ;

$include "dynamDef.gms"

loop(tt$(years(tt) le 2050),

   if (ord(tt) gt 1,

      ts(tt) = yes ;

$include 'iterloop.gms'

      options limrow=3, limcol=0 ;
      options solprint=off ;
      options iterlim=100 ;

      solve cge using mcp ;

      diagnostics("modStatus",ts) = cge.solvestat ;
      diagnostics("Walras",ts) = walras.l/inscale ;

      if (cge.solvestat eq 1,
         put screen ;
         put // "Solved for year ", years(tt):4:0 // ;
         tsim(ts) = yes ;
         loop(diag, put diag.tl:<10, diagnostics(diag,tt) / ; ) ;
         putclose screen ;
      else
         put screen ;
         put // "Failed to solve for year ", years(tt):4:0 // ;
         tsim(ts) = no ;
         loop(diag, put diag.tl:<10, diagnostics(diag,tt) / ; ) ;
         putclose screen ;

      ) ;

      ts(tt) = no ;

   ) ;

) ;

display diagnostics ;

execute_unload "%SIMNAME%.gdx" ;
```

Two other dynamic files are provided. One is called **DynNoShk.gms**. It is meant to test the consistency of the reference simulation. There are only two differences in this file compared to the **BaU.gms** file. The simulation flag *ifCal* is set to 0, and the output GDX filename is different. The output from this simulation should be identical to the output from the **BaU.gms** simulation. The other dynamic file is called **dynShock.gms**. It illustrates one way to implement a dynamic policy shock that is not part of the baseline. The flag *ifCal* is set to 0 and a new name for the GDX file is provided. The code extract below shows how the shock is implemented. The simulation reduces import tariffs by 50 percent. The shock is phased in linearly starting in 2015 and ending in 2025.

Listing 4.4: **Simulating a reduction in import tariffs**

```
$setGlobal REGION "ISO"
$setGlobal SIMNAME "dynShock"
$setGlobal BAUNAME "BaU"
```

```
*  Set ifCal to 1 for BaU scenarios

scalar ifCal / 0 / ;

$include "dynamDef.gms"

loop(tt$(years(tt) le 2050),

   if (ord(tt) gt 1,

      ts(tt) = yes ;

$include 'iterloop.gms'

      if(years(tt) gt 2014,

*        Reduce import tariffs between 2014 and 2025

         loop(t0,
            if(years(tt) le 2025,
               tm.fx(i, tt) =
                  tm.l(i, t0)*(1 + (0.5 - 1)*(years(tt) - 2014)/(2025-2014)) ;
            else
               tm.fx(i,tt) = 0.5*tm.l(i,t0) ;
            ) ;
         ) ;
      ) ;

      options limrow=3, limcol=0 ;
      options solprint=off ;
      options iterlim=100 ;

      solve cge using mcp ;

      diagnostics("modStatus",ts) = cge.solvestat ;
      diagnostics("Walras",ts) = walras.l/inscale ;

      if (cge.solvestat eq 1,
         put screen ;
         put // "Solved for year ", years(tt):4:0 // ;
         tsim(ts) = yes ;
         loop(diag, put diag.tl:<10, diagnostics(diag,tt) / ; ) ;
         putclose screen ;
      else
         put screen ;
         put // "Failed to solve for year ", years(tt):4:0 // ;
         tsim(ts) = no ;
         loop(diag, put diag.tl:<10, diagnostics(diag,tt) / ; ) ;
         putclose screen ;

      ) ;

      ts(tt) = no ;

   ) ;

) ;

display diagnostics ;

execute_unload "%SIMNAME%.gdx" ;
```

4.2.3 Post-simulation analysis

Post-simulation analysis is typically done directly for comparative static simulations. Essentially this involves including a GAMS file entitled **postsim.gms** that includes outputting a number of solution statistics—including optionally the SAM. Because of the structure of the file **postsim.gms**, it must be embedded in a loop over the set *sim*. In the case of a comparative static exercise, the set normally has a size of 1, with an appropriate label. The output of **postsim.gms** are two CSV files that can be directly read into Excel. One of the CSV files contains the post-simulation SAMs and the output of this file is optional.

[To be completed.]

Post-simulation of dynamic scenarios is somewhat more complicated. The solution of each dynamic simulation is saved as a GDX file. This is a compact way to save solutions across simulations. A separate GAMS file can be used to read multiple solution files and concatenate them together in a single CSV file. One example is provided in the file **maketab.gms**. From the user perspective there are only three modifications that need to be done to the file **maketab.gms**. The first is the names of the GDX files that will be merged. These must be given in the set defined by the name *sim*. The second is the name of the two output files. One file will contain the detailed results of all simulation listed in the set *sim*. This file will be a CSV file, potentially very large as the complete results of all of the simulations are merged together in one CSV file. The other file will contain the merged SAMs from the simulation. The latter is optional and determined by the setting of the flag *ifSAM*.

The **postsim.gms** file is not very sophisticated at the moment. Future enhancements will include the ability to filter results by variable, sector, time etc. in order to limit the size of the output and focus on some key results. The other will be to add code that allows for aggregating results—across sectors for example.

Two Excel files show how the results are read in to an Excel Pivot table and the creation of customized tables that query the Pivot table directly. One could also design customized charts from the same data source. The file **BaUScenarios.xlsx** shows some of the key results from the reference scenario. The file **nrgTax.xlsx** compares some of the key results from the subsidy elimination scenario with the baseline scenario.

[To be completed.]

4.3 Data preparation

There are a number of ways the user can modify the model's database. Most of this has been encapsulated in an Excel file—though it still in a beta phase—that will write-out all of the appropriate input files for the simulation. This is done via a VBA macro. Given the general reliability of Microsoft's products, there is no guarantee that the VBA macro will work correctly on any machine other than the one it was developed on! User's always have the option of directly editing any of the provided input files that have been written out with the VBA macro.

As mentioned above, there are two key data files that are rarely modified by the users—the base year SAM and the exogenous dynamic trends. The base year SAM is nonetheless part of the Excel package and will be written out by the VBA macro so the user has some control over this. And in the future, we may wish to add additional exogenous dynamic trends to the existing set including those from the national authorities.

The main user inputs are:

- Aggregation mappings for activities and commodities
- Definitions of subsets
- Key substitution, transformation and income elasticities
- Dynamic time framework
- Dynamic assumptions including choice of exogenous scenario

The following sections will describe each of the separate worksheets of the supplied Excel user input file.

Options

The *Options* worksheet contains most of the main options. The cells in light pink can be changed by the user. There is also a button on this worksheet that will run the macro when it is pushed with a mouse click. The main options are the following:

Activities

The *Activities* worksheet contains the aggregation mapping for the activities of the SAM. By default there is a one-to-one mapping between the activities of the full SAM and the model dimensions. [Given the tentative nature of

Table 4.3: Main options in the *Options* worksheet

Name	**Description**
Working directory	Working directory to which all data files will be written to
Language	Set the Boolean cell to `FALSE` for English, otherwise to `TRUE` for alternative or default language
Name of SAM file	Name of file that will contain base year SAM
Name of bridge file	Name of file that will contain aggregation mappings and all other user inputs
Name of scenario file	Name of file that will contain main inputs for dynamic scenario
Description	Description of SAM (in both the default language and English)
Scale factor for SAM	Input data needs to be scaled for model convergence purposes. GDP should be scaled so that it is around 100
Scale factor for population	Population should be scaled so that it is around 100
Population scenario	There are six population scenarios labeled SSP1 through SSP5 and UNMED
GDP Scenario	There are five GDP scenarios labeled SSP1 through SSP5
GDP model	There are two sets of GDP scenarios—OECD and IIASA
Scenario time framework	The time framework is specified in this part of the worksheet. It must start with the base year (2007) and end with the year 2100 or less. Any sequence of years works—with both annual and multi-annual gaps. N.B. The macro that writes the data files relies on a named range called *timeScen* to write out the time framework. If the user changes the number of time steps, the named range *timeScen* must be adjusted. This can be done by going to the `Formulas|Name Manager` tab in Excel and re-specifying the *timeScen* range.

this file, it is not recommended to change the mapping for the moment.] Beyond the mapping, the user can also specify labels and descriptors in both English and an alternative language, also called the default language (typically English). The descriptions and labels thus have four columns. The first and third columns (the descriptions and corresponding labels) are in the default language, and the second and fourth columns are in English. Finally the worksheet contains some subset definitions that are potentially used by the model. There are currently four activity subsets that are self-explanatory—*agr* (agriculture), *man* (manufacturing), *srv* (services), and *nrg* (energy). The subsets are defined with Boolean values. The last column, labeled *acal*, is active in dynamic scenarios. All activities that are part of the subset *acal* are assumed to be subject to the economy-wide productivity growth factor that is calibrated in the reference scenario.

Commodities

The *Commodities* worksheet contains the aggregation mapping for the commodities of the SAM. By default there is a one-to-one mapping between the commodities of the full SAM and the model dimensions. [Given the tentative nature of this file, it is not recommended to change the mapping for the moment.] Beyond the mapping, the user can also specify labels and descriptors in both the default language and English. There is a single subset of commodities for the energy commodities labeled *e*. This worksheet contains an additional mapping for household consumption. Household consumption is not defined on the basis of all commodities, but on an aggregation of commodities and uses a nested structure to determine the full demand for commodities.[6] Under the list of commodities is an additional set of labels and descriptors that determine the number of household commodities. One of the these will (and must) be an energy commodity and this needs to be indicated with a Boolean value. This is important because the model decomposes the energy bundle in a specific manner that is intended to capture inter-energy substitution. The right most columns of the commodity mapping then provide a mapping from the full list of commodities to the household commodities. Unlike a true transition matrix—a supplied commodity can be mapped only once to a consumed commodity.

Other

The *Other* worksheet contains information on the other SAM accounts, i.e. all of those that are neither activities nor commodities. Users are free to change the labels and the descriptors, but not the number of accounts. There are a number of subsets associated with these accounts that the user might change—although with extreme caution. These are listed below:

Table 4.4: Modifiable user sets

Subset	Description
fd	Domestic final demand agents
h	Households
f	Other domestic final demand agents
fp	Factors of production
lab	Labor types
ul	Unskilled labor types
cap	Capital account
lnd	Land account
nrs	Natural resource account
inst	Institutional accounts
entr	Enterprise accounts
labels	Used by the model, not to be changed
subset	Used by the model, not to be changed

Macro data

The *Macrodata* worksheet contains base year information on population (by broad cohort), and various exchange rates for the base year. It also contains a parameter that is used to calibrate the base year capital stock. It is the average gross return on capital. The aggregate base year capital stock is then set equal to the sum across all activities of payment to capital divided by the gross rate of return. Thus if the total capital remuneration is 1 700, and there is an estimate that the average gross rate of return is 17 percent, the capital stock will be set to 10 000. There may be independent estimates of the gross capital stock. The number entered in this worksheet should be consistent with that independent estimate and the level of gross capital remuneration from the SAM.

Production elasticities

The user enters activity related input parameters in the *ProdElas* worksheet. The first column, *oldShr*, is used only to calibrate the comparative static simulation. It contains an estimate of the share of *Old* capital in sectoral production. In effect it is used to weight the *Old* and *New* production elasticities in the calibration of the comparative static version of the model. Hence if the *Old* elasticity is 0 and the *New* elasticity is 1 and the share of *Old* capital is

[6] Ideally, a full transition matrix approach would be used to map household demand to supplied commodities. The approach taken here is a hybrid approach.

assumed to be 0.8, the weighted elasticity will be 0.2 ($0.8 \times 0 + 0.2 \times 1$). The following table describes the remaining columns that are mainly elasticities. Many must be specified by vintage type.[7]

Table 4.5: User-specified production elasticities

Name	Description
`sigmanr0`	Top level CES elasticity between natural resource factor and net output bundle
`sigmap0`	CES elasticity between intermediate demand and value added (plus energy)
`sigmav0`	CES elasticity across factors of production (including energy)
`sigmaul0`	CES elasticity across unskilled labor types
`sigmasl0`	CES elasticity across skilled labor types
`sigmak0`	CES elasticity between capital and energy
`sigmaks0`	CES elasticity between capital and skilled labor bundle
`sigmakb0`	CES elasticity between mobile and fixed capital (normally zero)
`sigmae0`	Inter-energy CES elasticity
`sigman0`	Non-energy intermediate substitution elasticity
`chiKF0`	Share of capital in base SAM to allocate to fixed capital (normally zero)
`rkf0`	Base year return to fixed capital (normally 1)
`omeganr0`	Supply elasticity of sector specific factor (i.e. natural resource)
`omegap0`	Supply transformation elasticity of make matrix
`invElas0`	Dis-investment elasticity for sectors in decline

CommElas

The user enters commodity related input parameters in the *CommElas* worksheet. The following table describes these that are all elasticities.

Table 4.6: User-specified commodity elasticities

Name	Description
`sigmas0`	The CES aggregation elasticity for the make matrix
`sigmam0`	CES (Armington) elasticity between domestic and imported goods
`sigmax0`	CET transformation elasticity between domestic and export markets
`sigmamg0`	The substitution elasticity for the 'production' of trade and transport services
`omegam0`	Import supply elasticity (normally infinity)
`etae0`	Export demand elasticity (normally infinity)

Final demand elasticities

The user enters final demand related input parameters in the *FDElas* worksheet. There are two tables of elasticities. The top table contains the following:

The second table contains the income and price elasticities used to calibrate the household demand function. Note that it is defined at the level of household commodities (indexed by k) and not at the level of all commodities.[8]

Dynamics

The *Dynamics* worksheet contains the default assumption regarding productivity and depreciation. There are six columns that apply to the model's activities.

7 The input vectors `sigmakb0`, `chiKF0` and `rfk0` were introduced for version 2.0c of the MANAGE model.

8 There are instances of SAM's where some pass-through institutions, such as an 'Enterprise' account consume goods and services. The model assumes that only Armington agents purchase goods and services. One possible solution is to treat these pass-through accounts as a household.

Table 4.7: User-specified final demand elasticities

Name	Description
`sigmac0(h,k)`	Energy-non energy substitution for good k across households (used in the consumer demand transition matrix)
`sigmacaa0(h,k)`	Substitution across non energy demand for good k and household h (used in the consumer demand transition matrix)
`sigmacae0(h,k)`	Substitution across energy demand for good k and household h
`sigmaf0(f)`	Substitution elasticity between energy and non energy in other final demand
`sigmafaa0(f)`	Substitution across non energy demand for other final demand
`sigmafae0(f)`	Substitution across energy demand for other final demand
`epsL0(l)`	Aggregate labor supply elasticity by skill
`omegaL0(l)`	Labor transformation elasticity by skill
`epsK0(vint)`	Aggregate capital supply elasticity (only used in comparative static model)
`omegaK0(vint)`	Capital transformation elasticity (only used in comparative static model)
`omegatl0(vint)`	Aggregate land supply elasticity
`omegat0(vint)`	Land transformation elasticity

The remaining two parameters in the worksheet relate to capital depreciation on the one hand (*depr*) and energy efficiency improvement (*AEEI*) for other domestic agents (e.g. households).

The *WPrice* worksheet contains assumptions about the exogenous world import and export prices.[9] This would normally be used to input price assumptions about key sectors such as energy and agriculture. The worksheet is split into two panels. The left-panel has the exogenous assumptions about import prices. These should be entered in level (or index) terms for all years. The model treats them as indices (i.e. only uses the implied growth rates). The second panel contains the assumptions about export prices.

Standard distribution files

The model is ready for immediate use. Users can start with running the comparative static model (**comp.gms**). The results will be stored in two CSV files—**comp.csv** and **compsam.csv** that can be read directly into Excel, or read into an Excel pivot table. The file **comp.xlsx** shows an example of the latter with some pre-formatted tables.

The baseline simulation (**bau.gms**) needs to be run before any other dynamic simulation because it calibrates some key variables needed for subsequent dynamic simulations. The two other dynamic simulations (**DynNoShk.gms** and **dynShock.gms**) can be run in any order afterwards. The file **maketab.gms** merges the results from the baseline and tariff reduction simulation and can be used as input to an Excel file.

The user can of course make changes to the input files. This can be done indirectly through the Excel file named **ISOBridge.xlsm** (in which case the VBA macro needs to be run), or directly by editing one of the two input files (**ISOBase.dat** and **ISObauScn.dat**).

The following table lists the files that are part of the standard distribution package—organized by cluster.

9 This worksheet was introduced with version 2.0c of the model.

Table 4.8: User-specified dynamic assumptions

Name	Description
`exProd`	The model allows for an exogenous and uniform shift in each activities production possibilities frontier and this will be captured in the `exProd` parameter that is used to update the `lambdan` and `lambdav` parameters in the model. It is specified as a percent increase per year.
`alphaL, betaL`	There is an economy-wide (labor) productivity shifter used to calibrate to a given GDP scenario in the baseline scenario. It is made sector-specific by a linear transformation, where `alphaL` is an additive shifter and `betaL` is a multiplicative shifter. Thus if `alphaL` is 0 and `betaL` is 1 for services, and `alphaL` is 2 and `betaL` is 1 for manufacturing, then productivity in manufacturing will be 2 percentage points higher than in services. If on the other hand `alphaL` is 0 and `betaL` is 2 in manufacturing, then productivity in manufacturing will be twice as high than in services—no matter what the level is in services. This can have unintended consequences. If services productivity is negative, it would imply that manufacturing productivity would be twice as negative.
`energyEff`	The energy efficiency parameter provides the improvement of energy use in each activity (across all energy carriers). It is specified as a percent increase per year. Caution should be taken in transformation sectors such as chemicals and petroleum refineries—as a large share of the energy inputs in these sectors may not be burned as energy but transformed into other products such as plastics and distillates.
`yexo`	This parameter is typically used to implement an autonomous change in yields. It is an efficiency factor linked to land use. A value of 1.5, for example, will increase the land productivity factor by 1.5 percent per annum.
`kprod`	This parameter is typically used to implement an autonomous change in capital productivity. It is an efficiency factor linked to capital use. A value of 1.5, for example, will increase the capital productivity factor by 1.5 percent per annum. It can be used, for example, in the livestock sector to capture exogenous improvements in the offtake rate in meat production.

Table 4.9: Files in the standard distribution package

Name	Description
Core data files	
ISOSAMv9.dat	Data file containing full SAM and full SAM labels
ISOBase.dat	Data file containing aggregation mappings, subsets, key model parameters
compscn.dat	Dummy data file needed to run comparative static model with 'dynamic' parameters
ISOBaUScn.dat	Exogenous data file with various population and GDP scenarios for the period 2007 through 2100
Excel data preparation file	
ISOBridge.xlsm	Excel file that can be used to prepare the core data files (see above). It encapsulates all the user input and a VBA macro can be implemented to create the core data files. It is still in a beta version so caution must be used in changing its structure.
Core model files	
cal.gms	GAMS code that initializes model variables and calibrates model parameters.
model.gms	GAMS code that contains model description including variable and parameter declarations and equation specifications.
iterloop.gms	GAMS code that is used between simulation periods to update exogenous variables and provide starting values for endogenous variables.
Simulation files	
comp.gms	Standard and basic comparative static simulation file. Normally used for diagnostic testing of the model.
dynamDef.gms	Common code for dynamic simulation files.
bau.gms	Baseline dynamic simulation file. Normally used to calibrate productivity.
dynNoShk.gms	Baseline dynamic simulation file. Normally used for dynamic diagnostic testing of the model. Its output should be identical to the standard baseline simulation output.
dynShock.gms	Dynamic simulation file that reduces import tariffs by 50 percent between 2014 and 2025.
Post-simulation files	
postsim.gms	GAMS code that saves key simulation results into a CSV file.
Maketab.gms	GAMS code that will merge multiple dynamic GDX output files into a single CSV file.

Appendix A

The CES and CET functions

This appendix describes in full detail the two functional forms most widely used in CGE models-the constant-elasticity-of-substitution (CES) and constant-elasticity-of-transformation (CET) functions. CES functions are widely used in demand functions where substitutability across different products and/or factors is needed and where the main objective is to minimize cost. CET functions are broadly used to determine supply functions across different markets where the main objective is to maximize revenues. The two are very similar in many ways and the algebraic derivations below will be more detailed for the CES function.

A.1 The CES function

A.1.1 Basic formulas

In production, the CES function is used to select an optimal combination of inputs (either goods and/or factors) subject to a CES production function. In consumer demand, the CES is used as a utility (or sub-utility) or preference function. In either case, the purpose is to minimize the cost of purchasing the 'inputs' subject to the production or utility function. In generic terms the system takes the following form:

$$\min_{X_i} \sum_i P_i X_i$$

subject to the constraint:

$$V = A\left[\sum_i a_i(\lambda_i X_i)^\rho\right]^{1/\rho}$$

The objective function represents aggregate expenditure. The constraint expression will be referred to as the CES primal function. The parameter A is an aggregate shifter that can be used to shift the overall production function (or utility function). Each input, X_i, is multiplied by an input-specific shifter, λ_i, that can be used to implement input-specific productivity increases (for example biased technological change), or specific changes in consumer preferences. The (primal) share coefficients, a_i, are typically calibrated to some base year data and held fixed. The CES exponent, ρ, is linked to the curvature of the CES function (and will be explained further below). For given component prices, P_i, and a given level of production or utility V, solving the optimization program above will yield optimal demand functions for the inputs, X_i.

The Lagrangian can be set up as:

$$\mathcal{L} = \sum_i P_i X_i + \Lambda\left(V - A\left[\sum_i a_i(\lambda_i X_i)^\rho\right]^{1/\rho}\right)$$

Taking the partial derivative with respect to X_i and the Lagrange multiplier Λ yields the following system of equations:

$$P_i = \Lambda a_i \lambda_i^\rho X_i^{\rho-1} A\left[\sum_i a_i(\lambda_i X_i)^\rho\right]^{(1-\rho)/\rho} = \Lambda a_i A^\rho \lambda_i^\rho X_i^{\rho-1} V^{1-\rho}$$

$$V = A\left[\sum_i a_i(\Lambda_i X_i)^\rho\right]^{1/\rho}$$

Taking the first expression, it can be multiplied by X_i, and then summed. This of course is equal to the value of the bundle, i.e. $P.V$, where P is the aggregate price:

$$P.V = \sum_i P_i X_i = \Lambda V^{1-\rho} A^\rho \sum_i a_i \lambda_i^\rho X_i^\rho = \Lambda V^{1-\rho} V^\rho = \Lambda V$$

This shows that Λ, the Lagrange multiplier is the same as the aggregate price, P. We can re-arrange expression above to get an expression for optimal input demand, where Λ is replaced by P:

$$X_i = a_i^{1/(1-\rho)} A^{\rho/(1-\rho)} \left(\frac{P}{P_i}\right)^{1/(1-\rho)} \lambda_i^{\rho/(1-\rho)} V$$

We finally end up with the following expression, where the CES primal exponent, ρ, is replaced by the so-called CES elasticity of substitution, σ:

$$X_i = \alpha_i (A\lambda_i)^{\sigma-1} \left(\frac{P}{P_i}\right)^\sigma V \tag{A.1}$$

where we made the following substitutions:

$$\sigma = \frac{1}{1-\rho} \Leftrightarrow \rho = \frac{\sigma-1}{\sigma} \Leftrightarrow \frac{\rho}{1-\rho} = \sigma - 1 \Leftrightarrow \rho.\sigma = \sigma - 1$$

and

$$\alpha_i = a_i^{1/(1-\rho)} = a_i^\sigma \Leftrightarrow a_i = \alpha_i^{1/\sigma}$$

Abstracting from the technology parameters, the demand equation implies that demand for 'input' X_i is a (volume) share of total demand V. The share, with equal prices is simply equal to α_i. With a positive elasticity of substitution, the share is sensitive to the ratio of prices relative to the aggregate price index. Since the component price is in the denominator, the demand for that component declines if its price rises relative to the average and vice versa if its price declines vis-à-vis the average price. The α parameters will be referred to as the CES dual share parameters (for reasons described below), and the a parameters are the primal CES share parameters. Notice that expression (A.1) simplifies if it is expressed in terms of efficiency inputs, X^e and efficiency prices, P^e:

$$X_i^e = \alpha_i A^{\sigma-1} \left(\frac{P}{P_i^e}\right)^\sigma V$$

where

$$X_i^e = \lambda_i X_i$$

and

$$P_i^e = \frac{P_i}{\lambda_i}$$

The aggregate price P can be determined using two expressions. The first is the zero profit condition:

$$P = \frac{\sum_i P_i X_i}{V}$$

The other is by inserting the optimal demand relation X_i (equation A.1) in the zero profit condition :

$$P.V = \sum_i P_i X_i = A^{\sigma-1} \sum_i P_i \alpha_i \left(\frac{P}{P_i}\right)^\sigma \lambda_i^{\sigma-1} V = P^\sigma A^{\sigma-1} V \sum_i \alpha_i \left(\frac{P_i}{\lambda_i}\right)^{1-\sigma}$$

The V's cancel out, and the aggregate price can then be expressed by the following formula:

$$P = \frac{1}{A}\left[\sum_i \alpha_i \left(\frac{P_i}{\lambda_i}\right)^{1-\sigma}\right]^{1/(1-\sigma)} = \frac{1}{A}\left[\sum_i \alpha_i \left(P_i^e\right)^{1-\sigma}\right]^{1/(1-\sigma)} \tag{A.2}$$

This is sometimes referred to as the dual price expression. It has virtually the same functional form as the CES primal, which is a CES aggregation of the input volumes using the primal share parameters as weights. The CES dual price formula is a CES aggregation of the input prices using the CES dual share parameters as weights and a different exponent. In a CGE model, the zero-profit condition or the dual price formula can be used interchangeably (with the proviso that the substitution elasticity differs from 1).[1] There is a simple formula for the budget shares given by:

$$s_i = \frac{P_i X_i}{P.V} = \alpha_i (A\lambda_i)^{\sigma-1} \left(\frac{P}{P_i}\right)^{\sigma} V \left(\frac{P_i}{P}\right) \frac{1}{V} = \alpha_i (A\lambda_i)^{\sigma-1} \left(\frac{P}{P_i}\right)^{\sigma-1} \tag{A.3}$$

Notice that this expression for the budget shares is only a function of prices. With the technology parameters set to 1, this simplifies further to:

$$s_i = \alpha_i \left(\frac{P}{P_i}\right)^{\sigma-1}$$

It turns out that the parameter σ measures the elasticity of substitution for the CES function and is constant over the entire domain. The elasticity of substitution is an indication of the curvature of an isoquant, see Varian (1992), i.e. it measures the rate of change of the ratio of inputs (in a 2-input case), relative to the change in their relative prices. For example, if the CES combines capital and labor to form output, a large substitution elasticity suggests that the factor proportions will change rapidly as one of the inputs becomes cheaper relative to the other. There are two limiting cases of interest. If the substitution elasticity is zero, then there is no substitution across inputs and the optimal choice is to use them in fixed proportion. At the other extreme, if the substitution elasticity is infinite, this is equivalent to saying the inputs are identical, and in this case, in equilibrium, the two inputs would have the same price. This could potentially be the case for electricity production. If there is a regional or national buyer of electricity, the buyer is most likely indifferent about how the electricity is produced and thus will purchase from the lowest cost producer (a perhaps somewhat simplified view of electricity markets.) This implies that the cost of the electricity inputs, from all sources (e.g. thermal, nuclear, etc.) would be (nearly) identical.

The elasticity of substitution across inputs is defined by the following formula:

$$\sigma = \frac{\partial\left(\frac{X_i}{X_j}\right)}{\partial\left(\frac{P_i}{P_j}\right)} \frac{\left(\frac{P_i}{P_j}\right)}{\left(\frac{X_i}{X_j}\right)}$$

The ratio of the optimal inputs using expression (A.1) is:

$$\frac{\alpha_i}{\alpha_j}\left(\frac{P_i}{P_j}\right)^{-\sigma}\left(\frac{\lambda_i}{\lambda_j}\right)^{\sigma-1}$$

Taking the partial derivative of the expression with respect to the ratio P_i/P_j and multiplying it by the second term of the elasticity of substitution yields the conclusion that the substitution elasticity is $-\sigma$. It is logical that it is negative. If the price of one input increases, say labor, relative to the other, say capital, producers would substitute away from labor towards capital, i.e. the ratio of labor to capital would drop as the price of labor increases relative to capital. Varian (1992) in fact defines the elasticity of substitution in terms of the absolute value of the technical rate of substitution, that measures the slope of the budget line. Numerically what it represents is the relative change in the ratios. If σ is 1, for example, and the price of labor increases by 10 percent relative to capital, the labor to capital ratio would decrease by (around) 10 percent.[2] The higher is σ, the more the proportion changes.

A.1.2 Special cases

There are three special cases that require additional derivations due to numerical restrictions on the primal and dual exponents. A substitution elasticity of 0 is clearly a special case and is referred to as a Leontief technology. From the dual price formula, it is clear that σ equal to 1 is a special case and is known as a Cobb-Douglas technology (or utility function). Finally, a value of ρ equal to 1 corresponds to infinite substitution elasticity and a linear primal aggregation function. This is also referred to as a case of perfect substitution.

[1] We shall see below that when the substitution elasticity is 1, both primal and dual expressions take a different functional form.

[2] The elasticity is a marginal concept that holds only approximately for large changes.

The Leontief case

The first special case is for the so-called Leontief functional form.[3] In this case the substitution elasticity is 0 and corresponds to a value for ρ that is $-\infty$. In this case the optimization program takes the following form:[4]

$$\min_{X_i} \sum_i P_i X_i$$

subject to the constraint:

$$V = \min\left(\frac{a_i}{\lambda_i X_i}\right)$$

The visual implementation has L-shaped isoquants. The Leontief technology constraint or production/utility function is discontinuous. Fortunately, the optimal demand functions are easy to implement and are just special cases of expression (A.1):

$$X_i = \frac{\alpha_i}{\lambda_i}\frac{V}{A}$$

$$P = \frac{1}{A}\sum_i \alpha_i \left(\frac{P_i}{\lambda_i}\right)$$

Thus the Leontief specification implies that inputs are always in fixed proportion relative to output and the aggregate price is simply the linear weighted aggregation of the input prices, where the weights are given by the input-output coefficients, adjusted by changes in efficiency. The efficiency parameter has a nice intuitive interpretation in this case. Say λ increases by 10 percent, then demand for the input declines by 10 percent.

The Cobb-Douglas function

Another special case is the so-called Cobb-Douglas function, very frequently used in introductory text books in microeconomics. The Cobb-Douglas function has a substitution elasticity of 1 implying that ρ is equal to 0. Clearly, this creates a problem for specifying the CES primal function as well as the CES dual price function. As with the Leontief, the optimal demand conditions are given by expression (A.1), with σ set to 1:

$$X_i = \alpha_i \left(\frac{P}{P_i}\right) V \Leftrightarrow s_i = \frac{P_i X_i}{P.V} = \alpha_i$$

The Cobb-Douglas specification has constant budget shares irrespective of relative prices (and changes in technology). Another implication of the Cobb-Douglas specification is that the dual shares must add up to 1 as they are equivalent to the budget shares. By definition, as well, the primal and dual shares are the same. The Cobb-Douglas primal and dual price functions have the following expressions:

$$V = A\prod_i (\lambda_i X_i)^{\alpha_i}$$

$$P = \frac{1}{A}\prod_i \left(\frac{P_i}{\alpha_i \lambda_i}\right)^{\alpha_i}$$

Rather than code the Cobb-Douglas function as a special case, many modelers choose to replace the elasticity of 1 with a value close to 1 such as 1.01. This would have only marginal repercussions on the results.

Perfect substitution

The third special case is for a substitution elasticity of infinity. In this case ρ takes the value of 1 and the primal function is a straight linear aggregation of the inputs. The optimal demand conditions cannot be used in the case of an infinite substitution elasticity. In its stead, the optimal demand condition is replaced with the law-of-one-price, adjusted by efficiency differentials, and the zero profit condition is replaced with the CES primal function, i.e. the linear weighted aggregation of the inputs:

$$\frac{P_i}{\alpha_i \lambda_i} = P$$

[3] Leontief, winner of the 1973 Nobel prize in Economics, is renowned for his work on input-output tables, much of which focused on fixed input technologies (!!!! reference).

[4] !!!! need a reference

$$V = \sum_i \alpha_i \lambda_i X_i$$

The aggregation function can be replaced by the zero profit condition:[5]

$$P.V = \sum_i P_i X_i$$

A.1.3 Calibration of the CES function

Standard calibration

Calibration typically involves inverting functional forms to evaluate the value of a parameter given initial values for variables. Prices and volumes, P_i, X_i, V and P, are normally initialized to a given database or SAM. This may or may not include actual price/volume splits. If not, prices will typically be initialized at unit value—potentially adjusted for a price wedge such as a tax or a margin. The substitution elasticities are also normally inputs—either derived from econometric estimation, other data bases or models, or from a literature review. This leaves the parameters λ_i, α_i and A to calibrate. The technology parameters are normally associated with dynamics, so there is little reason not to initialize them to unit value as they can be incorporated in the initial share parameter value without any loss in generality. Thus, the only parameters left to calibrate are the α_i from which it is possible to derive the primal share parameters, a_i, if needed. The calibration formula is derived from the inversion of equation (A.1):

$$\alpha_i = \left(\frac{X_i}{V}\right)\left(\frac{P_i}{P}\right)^{\sigma}(A.\lambda_i)^{1-\sigma} = \left(\frac{X_i}{V}\right)\left(\frac{P_i}{P}\right)^{\sigma}$$

The right-most term is the most used formula where the technology parameters are explicitly set to 1.[6]

An alternative calibration

An alternative, that is used in many CGE models, is to assume that the primal shares sum to 1.[7] In this case, the aggregate shifter, A, also needs to be calibrated and only exceptionally would be equal to 1. Using the definitions above, this implies the following restriction:

$$\sum_i a_i = \sum_i \alpha_i^{1/\sigma} = 1$$

Using equation (A.1), the restriction can be used to calibrate the A parameter (assuming the component specific technology shifters are equal to 1):

$$\alpha_i = A^{1-\sigma}\left(\frac{P_i}{P}\right)^{\sigma}\frac{X_i}{V}$$

$$\alpha_i^{1/\sigma} = A^{(1-\sigma)/\sigma}\left(\frac{P_i}{P}\right)\left(\frac{X_i}{V}\right)^{1/\sigma}$$

$$1 = \sum_i \alpha_i^{1/\sigma} = \frac{A^{(1-\sigma)/\sigma}}{P.V^{1/\sigma}}\sum_i P_i X_i^{1/\sigma}$$

The calibrated A parameter is then given by the following expression:

$$A = \left[\frac{P.V^{1/\sigma}}{\sum_i P_i X_i^{1/\sigma}}\right]^{\sigma/(1-\sigma)}$$

[5] Modelers have the choice of using the primal aggregation function or the revenue function. The latter holds in all three special cases for the substitution elasticity.

[6] In many introductions to CGE models, the calibration formulas explicitly exclude the price term. This is a dangerous practice that can lead to model bugs that can be hard to detect. It is best to explicitly initialize prices to 1 and use the correct calibration formula. In fact, one way to test model calibration and specification is to initialize prices to an arbitrary value and initialize volumes subject to these prices. Simulating a counter-factual with no shocks should replicate the initial data solution. If not, there is an error in initialization, calibration and/or specification.

[7] See for example Devarajan et al. (1997), Lofgren et al. (2002), Lofgren et al. (2013).

Inserting the expression for A, the α parameters can be calibrated using the following:

$$\alpha_i = \left[\frac{P.V^{1/\sigma}}{\sum_i P_i X_i^{1/\sigma}}\right]^{\sigma} \left(\frac{P_i}{P}\right)^{\sigma} \frac{X_i}{V} = \frac{P_i^{\sigma} X_i}{\left[\sum_i P_i X_i^{1/\sigma}\right]^{\sigma}}$$

A.1.4 Calibration example

The example will show the calibration of the Armington function. The Armington function combines domestically produced goods, XD, with imported goods, XM, to form the so-called Armington aggregate good, XA. Their respective prices are PD, PM, and PA. Assume that prices are initialized at 1 and XD and XM are respectively 80 and 20. Table A.1 shows the calibrated parameters under both calibration options with an elasticity of 2. With unitary prices the dual share parameters are equal to the budget shares when calibrating under option 1. The primal share parameters sum to 1 under option 2 (but are not equal to the budget shares).

Table A.1: Calibration example of Armington function with unitary prices

	Dual shares		Primal shares	
Shifter	Domestic	Imported	Domestic	Imported
1.0	0.8	0.2	0.8944	0.4472
1.8	0.4444	0.1111	0.6667	0.3333

Now, let's assume that the value shares are identical, but that the price of imports is no longer 1, but 1 plus an import tariff. Border prices are initialized at 1, but end-user prices are tariff inclusive. Assume that the tariff is 25%, than the volume of imports is not 20, but 16. The calibrated parameters then take the values in Table A.2. The dual share parameters no longer sum to unity, i.e. they are no longer equivalent to the budget shares.[8] The primal shares still sum to 1, by design, but are not the same as when all prices are unitary, nor do they line up with the budget shares.

Table A.2: Calibration example of Armington function with non-unitary prices

	Dual shares		Primal shares	
Shifter	Domestic	Imported	Domestic	Imported
1.0	0.8	0.25	0.8944	0.5
1.944	0.4114	0.1286	0.6414	0.3586

A.1.5 Alternative functional forms

In single country CGE models, the Armington specification could take the following form:

$$XD = \alpha^d A^{\sigma-1} XA \left(\frac{PA}{PD}\right)^{\sigma^m}$$

$$XM = \alpha^m A^{\sigma-1} XA \left(\frac{PA}{PM}\right)^{\sigma^m}$$

$$PA.XA = PD.XD + PM.XM$$

or

$$PA = \frac{1}{A}\left[\alpha^d PD^{1-\sigma^m} + \alpha^m PM^{1-\sigma^m}\right]^{1/(1-\sigma^m)}$$

8 The domestic share parameter is nonetheless equal to the budget share because both its price and the Armington price are still assumed to be unitary.

with the assumption that A is equal to 1 in the base year. Heuristically, one can think of these equations as determining XD, XM and PA, given the component prices, PD and PM, and the overall level of Armington consumption, XA. Many other modelers use the following set of equations to implement the Armington assumption (for example see Lofgren et al. (2013)):[9]

$$\frac{XM}{XD} = \frac{\alpha^m}{\alpha^d}\left(\frac{PD}{PM}\right)^{\sigma^m} = \left(\frac{a^m}{1-a^m}\frac{PD}{PM}\right)^{1/(1-\rho^m)}$$

$$XA = A[a^m XM^\rho + (1-a^m)XD^\rho]^{1/\rho}$$

$$PA.XA = PD.XD + PM.XM$$

One can think of the first equation as determining XM. The second equation determines XD as XA is exogenous in this system. And the third is the standard zero profit condition. Which set of equations one uses is irrelevant—the systems are identical. The first set has several attractive features. First, the entire system can be written strictly in terms of the substitution elasticity and the dual share parameters. There is no need to carry around the primal share parameters nor the primal exponent. Second, it more easily lends itself to generalization. Its generic formulation readily lends itself to more than two components and no restrictions are imposed on the parameters.

A.1.6 Normalized CES

It is sometimes the case that the CES is badly scaled, particularly when working with actual price/volume splits. One work-around is to normalize all variables such that they are all equal to 1 in the base case—$\bar{X} = X/X_0 = 1$, where X_0 is a base level. The CES equations become:

$$X_{i,0}\bar{X}_i = \alpha_i(A\lambda_i)^{\sigma-1}\left(\frac{P_0\bar{P}}{P_{i,0}\bar{P}_i}\right)^\sigma V_0\bar{V}$$

$$P_0\bar{P} = \frac{1}{A}\left[\sum_i \alpha_i\left(\frac{P_{i,0}\bar{P}_i}{\lambda_i}\right)^{1-\sigma}\right]^{1/(1-\sigma)}$$

The initial values can be collected to yield the following expressions:

$$\bar{X}_i = \alpha_i\left(\frac{V_0}{X_{i,0}}\left(\frac{P_0}{P_{i,0}}\right)^\sigma\right)(A\lambda_i)^{\sigma-1}\left(\frac{\bar{P}}{\bar{P}_i}\right)^\sigma \bar{V}$$

$$\bar{P} = \frac{1}{A}\left[\sum_i \alpha_i\left(\frac{P_{i,0}}{P_0}\right)^{1-\sigma}\left(\frac{\bar{P}_i}{\lambda_i}\right)^{1-\sigma}\right]^{1/(1-\sigma)}$$

Define the following parameter:

$$\chi_i = \frac{V_0}{X_{i,0}}\left(\frac{P_0}{P_{i,0}}\right)^\sigma = \frac{1}{s_{i,0}}\left(\frac{P_{i,0}}{P_0}\right)^{1-\sigma}$$

where s_i is the initial value share defined above. The demand expression can be converted to:

$$\bar{X}_i = \alpha_i\chi_i(A\lambda_i)^{\sigma-1}\left(\frac{\bar{P}}{\bar{P}_i}\right)^\sigma \bar{V}$$

Given the calibration formula, it is the case that $\alpha_i\chi_i = 1$. It may be nonetheless useful to carry both terms if technology or preference shifters are embodied in the α terms.

The price index expression can be written as:

$$\bar{P} = \frac{1}{A}\left[\sum_i s_{i,0}\alpha_i\chi_i\left(\frac{\bar{P}_i}{\lambda_i}\right)^{1-\sigma}\right]^{1/(1-\sigma)}$$

If we do not need to carry through the original calibrated share parameters, we can write the two expressions as:

$$\bar{X}_i = (A\lambda_i)^{\sigma-1}\left(\frac{\bar{P}}{\bar{P}_i}\right)^\sigma \bar{V}$$

[9] In the Lofgren et al. (2013) formulation, the primal exponent, ρ, has the opposite sign.

$$\bar{P} = \frac{1}{A}\left[\sum_i s_{i,0}\left(\frac{\bar{P}_i}{\lambda_i}\right)^{1-\sigma}\right]^{1/(1-\sigma)}$$

Both expressions are now only a function of the normalized variables and the base year shares. Any subsequent use of the variables requires re-scaling, for example $X_{i,0}\bar{X}_i$.

A.1.7 Comparative statics

Elasticities

This section will derive some of the key elasticities of the CES function. The first relationship is the elasticity of the aggregate price with respect to a component price:

$$\frac{\partial P}{\partial P_i}\frac{P_i}{P} = s_i = \frac{P_i X_i}{P.V}$$

The elasticity of the aggregate price relative to an input price is equal to the budget share, irrespective of the substitution elasticity. The matrix of own- and cross-price elasticities, holding the aggregate volume constant is given by the following formula:

$$\varepsilon_{ij} = \frac{\partial X_i}{\partial P_j}\frac{P_j}{X_i} = \sigma(s_j - \delta_{ij})$$

where δ_{ij} is the so-called Kronecker's delta that takes the value 1 for i equal to j, else it takes the value 0. Since σ is positive, all components are gross substitutes in the CES.

Example: Trade elasticities

In this section we derive the elasticity of demand for a good sourced in region s, using the standard dual nested Armington formulation. Let XA be the top level Armington demand for a commodity that is derived from a generic demand function—for example an LES, with an associated price PA. The Armington demand is split into two components, XD, a domestic good, and XMT, an aggregate (or composite import demand), with prices PD and PMT respectively. The aggregate import bundle, XMT, is broken out by region of origin to determine XM_s, import from region s, with a domestic import price of PM_s (i.e. the landed or CIF price tariff inclusive).

Given XA, the following equations determine the top-level demand system, where σ^m is the top-level substitution elasticity (i.e. between the domestic goods and aggregate imports):

$$XD = \alpha^d\left(\frac{PA}{PD}\right)^{\sigma^m} XA \tag{A.4}$$

$$XMT = \alpha^m\left(\frac{PA}{PMT}\right)^{\sigma^m} XA \tag{A.5}$$

$$PA = \left[\alpha^d PD^{1-\sigma^m} + \alpha^m PMT^{1-\sigma^m}\right]^{1/(1-\sigma^m)} \tag{A.6}$$

The next set of equations determines imports by region of origin, where σ^w represents the elasticity of substitution for imports across region of origin:

$$XM_s = \alpha_s^w\left(\frac{PMT}{PM_s}\right)^{\sigma^w} XMT \tag{A.7}$$

$$PMT = \left[\sum_s \alpha_s^w PM_s{}^{1-\sigma^w}\right]^{1/(1-\sigma^w)} \tag{A.8}$$

The elasticity we would like to derive is the following:

$$\varepsilon_s^w = -\frac{\partial XM_s}{\partial PM_s}\frac{PM_s}{XM_s} \tag{A.9}$$

In other words, by how much would demand for imports from region s change relative to a change in its own price (including possibly a reduction (or increase) in the applied tariff).

The starting point for the derivation is equation (A.7) from which it is possible to derive the following expression:

$$\begin{aligned}\varepsilon_s^w &= -\frac{\partial XM_s}{\partial PM_s}\frac{PM_s}{XM_s} = -\left[(1-s_s)\sigma^w - \frac{\partial XMT}{\partial PM_s}\frac{PM_s}{XMT}\right] \\ &= -\left[(1-s_s)\sigma^w + s_s\left(-\frac{\partial XMT}{\partial PMT}\frac{PMT}{XMT}\right)\right]\end{aligned} \tag{A.10}$$

Equation (A.10) shows that the trade elasticity is equal to the weighted average of the second-level Armington elasticity and the elasticity of demand for aggregate imports relative to the aggregate price of imports, where the weights are given by the import share of region s in total imports:

$$s_s = \frac{PM_s \cdot XM_s}{PMT \cdot XMT}$$

The final term in equation (A.10), not surprisingly takes the following form as derived from equation (A.5):

$$\begin{aligned}-\frac{\partial XMT}{\partial PMT}\frac{PMT}{XMT} &= -\left[(1-s_m)\sigma^m - \frac{\partial XA}{\partial PMT}\frac{PMT}{XA}\right] \\ &= -\left[(1-s_m)\sigma^m + s_m\left(-\frac{\partial XA}{\partial PA}\frac{PA}{XA}\right)\right]\end{aligned} \tag{A.11}$$

Similar to equation (A.10), equation (A.11) shows that the aggregate elasticity of import demand is the weighted average of the top-level Armington demand and the overall price elasticity of the (Armington) demand with respect to the (Armington) price, where the weight is given by the (aggregate) import share in total demand:

$$s_m = \frac{PMT \cdot XMT}{PA \cdot XA}$$

These equations use the following expressions that provide the elasticity of the composite CES price with respect to its individual price components:

$$\frac{\partial PMT}{\partial PM_s}\frac{PM_s}{PMT} = s_s$$

$$\frac{\partial PA}{\partial PMT}\frac{PMT}{PA} = s_m$$

Finally, the trade elasticity is given by the following expression:

$$\varepsilon_s^w = -\left[(1-s_s)\sigma^w + s_s\left((1-s_m)\sigma^m + s_m\epsilon^d\right)\right] \tag{A.12}$$

where

$$\varepsilon^d = -\frac{\partial XA}{\partial PA}\frac{PA}{XA}$$

Formulas in percent differences

It is useful in terms of comparative static analyses to convert the basic equations into percent differences. It is easy to trace out the impacts of a change in one of the 'exogenous' variables on demand and the overall price index. This is also the form of the equations used for models implemented in GEMPACK such as MONASH-style models.

The following expressions convey expressions (A.1) and (A.2) into their percent difference form:

$$\frac{\partial X_i}{X_i} = \dot{X}_i = \dot{V} + \sigma\left(\dot{P} - \dot{P}_i\right) + (\sigma - 1)\left(\dot{A} + \dot{\lambda}_i\right)$$

$$\frac{\partial P}{P} = \dot{P} = -\dot{A} + \sum_i s_i\dot{P}_i - \sum_i s_i\dot{\lambda}_i = -\dot{A} + \sum_i s_i\left(\dot{P}_i - \dot{\lambda}_i\right)$$

Thus the percent change in the unit cost, P, for a change in the input price, P_i, all else equal, is (approximately) the value share of component i—as already noted above.

A.1.8 Growth Accounting

Use can be made of the linearization above to derive the linearized growth accounting formula:

$$\frac{\Delta V}{V} = \frac{\Delta A}{A} + \sum_i s_i\frac{\Delta x_i}{x_i} + \sum_i s_i\frac{\Delta\lambda_i}{\lambda_i}$$

A.1.9 Parameter twists

The basic analytics

This final section on the CES describes how to adjust the share parameters in a dynamic scenario under a specific assumption—this is called the twist adjustment and is a core feature of the dynamic MONASH model, see Dixon and Rimmer (2002). The basic idea is to alter the share parameter, in a two-component CES, to target a given change in the ratio of the two components, however, with neutral impacts on the aggregate cost. For example, the target may be a cost-neutral increase in the capital/labor ratio by x%, or an increase in the import to domestic ratio of y%.

The ratio of the two components is given by the following expression using equation (A.1) as the starting point:

$$R = \frac{\alpha_1 \lambda_1^{\sigma-1} {P_2}^{\sigma}}{\alpha_2 \lambda_2^{\sigma-1} {P_1}^{\sigma}}$$

The idea is to move the initial ratio, R_{t-1} to R_t by tw percent.

$$\frac{R_t}{R_{t-1}} = (1 + tw_t)$$

Using the formulas above, we have:

$$\frac{R_t}{R_{t-1}} = (1 + tw_t) = \frac{\left(\frac{\lambda_{1,t}}{\lambda_{1,t-1}}\right)^{\sigma-1}}{\left(\frac{\lambda_{2,t}}{\lambda_{2,t-1}}\right)^{\sigma-1}} = \frac{(1+\pi_{1,t})^{\sigma-1}}{(1+\pi_{2,t})^{\sigma-1}}$$

The π variables represent the growth (either positive or negative) that will be applied to the technology parameters under the assumption of cost-neutral technological change. We can start with the dual cost function for year t, but with year $t-1$ prices:

$$\begin{aligned} P_{t-1}^{1-\sigma} &= \alpha_1 \left(\frac{P_{1,t-1}}{\lambda_{1,t}}\right)^{1-\sigma} + \alpha_2 \left(\frac{P_{2,t-1}}{\lambda_{2,t}}\right)^{1-\sigma} \\ &= \alpha_1 (1+\pi_{1,t})^{\sigma-1} \left(\frac{P_{1,t-1}}{\lambda_{1,t}}\right)^{1-\sigma} + \alpha_2 (1+\pi_{2,t})^{\sigma-1} \left(\frac{P_{2,t-1}}{\lambda_{2,t}}\right)^{1-\sigma} \end{aligned}$$

Recall that the share equation is given by:

$$s_{i,t-1} = \alpha_i \lambda_{i,t-1}^{\sigma-1} \left(\frac{P_t}{P_{i,t-1}}\right)^{\sigma-1}$$

Dividing through the expression above by $P_t^{1-\sigma}$ and inserting the share expressions for year $t-1$, we end up with:

$$1 = s_{1,t-1}(1+\pi_{1,t})^{\sigma-1} + s_{2,t-1}(1+\pi_{2,t})^{\sigma-1}$$

Solving in terms of π_1, we have:

$$(1+\pi_{1,t})^{\sigma-1} = \frac{1 - s_{2,t-1}(1+\pi_{2,t})^{\sigma-1}}{s_{1,t-1}}$$

and this can be inserted into the twist target formula to get:

$$1 + tw_t = \frac{1 - s_{2,t-1}(1+\pi_{2,t})^{\sigma-1}}{s_{1,t-1}(1+\pi_{2,t})^{\sigma-1}} = \frac{(1+\pi_{2,t})^{1-\sigma} - s_{2,t-1}}{s_{1,t-1}}$$

Finally, π_2 can be isolated to yield:

$$1 + \pi_{2,t} = \left[s_{1,t-1}(1+tw_t) + s_{2,t-1}\right]^{1/(1-\sigma)} = \left[1 + s_{1,t-1} tw_t\right]^{1/(1-\sigma)}$$

We can re-insert this into the expression above to derive an expression for π_1:

$$1 + \pi_{1,t} = \left[\frac{1 + s_{1,t-1} tw_t}{1 + tw_t}\right]^{1/(1-\sigma)}$$

Finally, the productivity update formulas that incorporate the twist adjustment take the form:

$$\lambda_{1,t} = (1+\pi_{1,t})\lambda_{1,t-1} = \left[\frac{1 + s_{1,t-1} tw_t}{1 + tw_t}\right]^{1/(1-\sigma)} \lambda_{1,t-1}$$

$$\lambda_{2,t} = (1+\pi_{2,t})\lambda_{2,t-1} = \left[1 + s_{1,t-1} tw_t\right]^{1/(1-\sigma)} \lambda_{2,t-1}$$

It is possible to generalize these formulas by partitioning the set of CES components into two sets—a set indexed by 1 that is the target set, and a set indexed by 2 that is the complement. For example, think of a set of electricity technologies that includes conventional and advanced. It is possible then to provide the same twist to all of the new technologies relative to the conventional technologies. The only change in the formulas above is that the share variable for the single component is replaced by the sum of the shares for the bundle of components:

$$\lambda_{1,t} = (1+\pi_{1,t})\lambda_{1,t-1} = \left[\frac{1+tw_t\sum_{i\in 1} s_{i,t-1}}{1+tw_t}\right]^{1/(1-\sigma)} \lambda_{1,t-1}$$

$$\lambda_{2,t} = (1+\pi_{2,t})\lambda_{2,t-1} = \left[1+tw_t\sum_{i\in 1} s_{i,t-1}\right]^{1/(1-\sigma)} \lambda_{2,t-1}$$

Converting to percent differences

The π factors reflect a percentage change in the relevant productivity factors for each of the components. Using a Taylor series approximation, the formulas above can be converted to a linear equation that is used by the Monash-style models. For the first component, we have:

$$\pi_1 = F(tw) = \left[\frac{1+s_1 tw}{1+tw}\right]^{1/(1-\sigma)} - 1 \approx F(0) + tw.F'(0) = -tw\frac{1-s_1}{1-\sigma}$$

For the second component we have:

$$\pi_2 = F(tw) = [1+s_1 tw]^{1/(1-\sigma)} - 1 \approx F(0) + tw.F'(0) = tw\frac{s_1}{1-\sigma}$$

Note that in the Monash models, the signs are reversed because the productivity factors divide the volume components whereas in the formulation above the productivity factors are multiplicative.

Examples of twisting the share parameters

We demonstrate these concepts with two examples. The first is a CES production function of capital and labor, where the labor share is 60% and the capital/labor substitution elasticity (i.e. σ) is set to 0.9. Prices are initialized at 1, therefore the original capital/labor ratio is 2/3. The target is to raise the capital/labor ratio 10% assuming cost neutrality. Table A.3 shows the key results. Labor efficiency would increase by 48% and capital efficiency would decline by 43%.

Table A.3: Example of capital/labor twist

	Labor	Capital	Capital/labor ratio
Initial	60.0	40.0	0.6667
After twist	57.7	42.3	0.7333
Percent change	-3.8	5.8	10.0
Growth factor	0.48	-0.43	

The second example comes from trade and the Armington assumption. Assume an 80/20 split between domestic goods and imports in value and volume implying a ratio of imports to demand of domestic goods of 0.25. Table A.4 shows the twist parameters needed to achieve an increase in this ratio of 10 percent with an Armington elasticity of 2. The preference parameter for imports increases by nearly 8 percent, while that for domestic goods decreases by 2 percent.

A.1.10 Summary

In summary, the CES functional form is often used as a production (or sub-production) function that combines two or more inputs to form output (or an intermediate composite bundle), under the assumption of cost minimization. It is also frequently used to maximize utility (or sub-utility) over a set of two or more goods, again with the assumption of cost minimization. Table A.5 highlights the two main expressions to emerge from the optimization—the derived demand functions, X_i, and the CES dual price expression, P. The top row shows the expression with all technology parameters initialized at 1, and the bottom row the most generic version.

Table A.4: Example of Armington import/domestic twist

	Domestic	Import	Import/domestic ratio
Initial	80.0	20.0	0.250
After twist	78.4	21.6	0.275
Percent change	-2.0	7.8	10.0
Growth factor	-0.02	0.08	

Table A.5: Key equations for CES implementation

	Demand	Aggregate price
Basic	$X_i = \alpha_i V \left(\frac{P}{P_i}\right)^{\sigma}$	$P = \left[\sum_i \alpha_i P_i^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$
with full technology	$X_i = \alpha_i (A\lambda_i)^{\sigma-1} V \left(\frac{P}{P_i}\right)^{\sigma}$	$P = \frac{1}{A}\left[\sum_i \alpha_i \left(\frac{P}{\lambda_i}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$

A.2 The CET Function

A.2.1 The basic formulation

This section describes the constant-elasticity-of-transformation (CET) function. The CET function is often used to describe a transformation frontier between two or more outputs. For example, a producer may produce two or more products and decides how much of each to produce based on market conditions, i.e. relative prices. The CET is often used to represent a producer's decision on the allocation of output between domestic and foreign markets. Another example is land supply, where land will be allocated across different uses according to the relative returns. The transformation elasticity is assumed to be uniform between any pair of outputs and therefore is analogous to the demand-based CES function described in detail above. The exposition of the CET will be much more succinct than that of the CES because most of the derivations can be derived in a similar fashion.

The CET can be setup as a revenue maximization problem, subject to a transformation frontier:

$$\max_{X_i} \sum_i P_i X_i$$

subject to

$$V = A\left[\sum_i g_i (\lambda_i X_i)^{v}\right]^{1/v}$$

where V is the aggregate volume (e.g. aggregate supply), X_i are the relevant components (sector-specific supply), P_i are the corresponding prices, g_i are the CET (primal) share parameters, and ν is the CET exponent. The CET exponent is related to the CET transformation elasticity, ω via the following relation:

$$\nu = \frac{\omega+1}{\omega} \Leftrightarrow \omega = \frac{1}{\nu - 1}$$

The transformation elasticity is assumed to be positive. Solution of this maximization problem leads to the following first order conditions:

$$X_i = \gamma_i (A\lambda_i)^{-1-\omega} \left(\frac{P_i}{P}\right)^{\omega} V \tag{A.13}$$

and

$$P = \frac{1}{A}\left[\sum_i \gamma_i \left(\frac{P_i}{\lambda_i}\right)^{1+\omega}\right]^{1/(1+\omega)} \tag{A.14}$$

where the γ_i parameters are related to the primal share parameters, g_i, by the following formula:

$$\gamma_i = g_i^{-\omega} \Leftrightarrow g_i = \left(\frac{1}{\gamma_i}\right)^{1/\omega}$$

From expression A.13, and ignoring the technology parameters for the moment, the clear difference with the CES expression for optimal demand (equation A.1) is that the component price is in the numerator and the aggregate price in the denominator. This is intuitively logical. If the supply price to a market goes up relative to the average market price, one would anticipate that supply would increase to that market. The greater the transformation elasticity the greater are the market shifts.

Calibration is similar to the CES case. Prices and volumes are initialized using base year data. Equation (A.13) can then be inverted to calculate the share parameters, γ_i, with typically the technology parameters initialized to the value 1. In most implementations, there is no need to carry around the primal share parameters, nor the primal exponent.

The main interesting case for the CET is the case of perfect transformation, i.e. the transformation elasticity is infinity. In this case the CET exponent is 0 and the aggregation function is a linear weighted aggregation of the components. The standard CET equations are then replaced by the law-of-one price and the linear aggregation function (or alternatively, the zero profit condition).

$$\frac{P_i}{A\lambda_i} = P \forall i$$

$$A \sum_i \lambda_i X_i = X$$

A.2.2 Converting to percent differences

It is easier to interpret or decompose the results of a simulation by looking at the CET equations in percent differences form—that is the standard form for MONASH-style models and implementation in GEMPACK. The following equations show the equations in percent difference form:

$$\frac{\partial X_i}{X_i} = \dot{X}_i = \dot{V} + \omega\left(\dot{P}_i - \dot{P}\right) - (\omega + 1)\left(\dot{\lambda}_i + \dot{A}\right)$$

$$\frac{\partial P}{P} = \dot{P} = -\dot{A} + \sum_i s_i \dot{P}_i - \sum_i s_i \dot{\lambda}_i = -\dot{A} + \sum_i s_i\left(\dot{P}_i - \dot{\lambda}_i\right)$$

where the variable s_i is the value share of component i in total revenue:

$$s_i = \frac{P_i X_i}{P \cdot V} = \gamma_i \left(\frac{P_i}{A \cdot \lambda_i \cdot P}\right)^{\omega+1}$$

A.2.3 Twists and the CET

It is relatively easy to show that the analogous twist formulas for the CET are:

$$1 + \pi_{1,t} = \left[\frac{1 + tw_t}{1 + s_{1,t-1} tw_t}\right]^{1/(1+\omega)} \qquad 1 + \pi_{2,t} = \left[\frac{1}{1 + s_{1,t-1} tw_t}\right]^{1/(1+\omega)}$$

A.3 The CRESH and CRETH functions

The CES and CET functions assume uniform pairwise substitution and transformation elastities. This section describes a generalization of these two functions that allow for pairwise specific elasticities.

A.3.1 The CRESH function

The *constant ratios of elasticities of substitution, homethetic* (CRESH) function is a generalization of the CES function that allows for pair-wise differences in the substitution elasticities. For example, assume that aggregate imports are allocated across regions of origins using the Armington assumption. If there is a single CES nest, this implies that the substitution across any two pairs of trading partners is identical, for example the substitution elasticity between imports from France and Germany is identical to the substitution elasticity between France and Thailand. The CRESH function allows to differentiate the substitution between any pair of inputs.

Theoretical derivation

The CRESH production function was developed by Hanoch (Hanoch (1971)) and has been used in applied general equilibrium models in the ORANI tradition (Dixon et al. (1982) and Dixon and Rimmer (2002)). The CRESH primal function (production and/or demand) starts with an implicit functional form:

$$F(X_i, V) = \sum_i \frac{c_i}{\rho_i} \left(\frac{A\lambda_i X_i}{V} \right)^{\rho_i} \equiv K \tag{A.15}$$

using the same notation as above where X_i are the inputs, λ_i are technology shifters and V is output. In this formulation K is a calibrated parameter. We can see that if the exponents, ρ_i, are identical across all i, the CRESH function becomes the CES function, where the primal parameters are given by $a_i = c_i/(K\rho)$.

Cost minimization subject to the production constraint above yields the following Lagrangian:

$$\mathcal{L} = \sum_i P_i X_i - \Lambda \left(\sum_i \frac{c_i}{\rho_i} \left(\frac{A\lambda_i X_i}{V} \right)^{\rho_i} - K \right)$$

Taking the partial derivative with respect to X_i and the Lagrange multiplier Λ yields the following system of equations:

$$P_i = \Lambda c_i X_i^{\rho_i - 1} \left(\frac{A\lambda_i}{V} \right)^{\rho_i} \tag{A.16}$$

$$\sum_i \frac{c_i}{\rho_i} \left(\frac{A\lambda_i X_i}{V} \right)^{\rho_i} \equiv K \tag{A.17}$$

The first equation can be used to isolate X_i:

$$X_i = \left(\frac{c_i \Lambda}{P_i} \right)^{\sigma_i} \left(\frac{V}{A\lambda_i} \right)^{1-\sigma_i} = c_i^{\sigma_i} \left(A\lambda_i \right)^{\sigma_i - 1} \left(\frac{1}{P_i} \frac{\Lambda}{V} \right)^{\sigma_i} V \tag{A.18}$$

where we have the same relations as above between the primal exponents (ρ) and the dual exponents (σ):

$$\sigma_i = \frac{1}{1-\rho_i} \Leftrightarrow \rho_i = \frac{\sigma_i - 1}{\sigma_i} \Leftrightarrow \frac{\rho_i}{1-\rho_i} = \sigma_i - 1 \Leftrightarrow \rho_i.\sigma_i = \sigma_i - 1$$

Given the form above that looks similar to the first order conditions for the CES, it appears natural to define a variable P^c to replace the ratio Λ/V:

$$X_i = \alpha_i \left(A\lambda_i \right)^{\sigma_i - 1} \left(\frac{P^c}{P_i} \right)^{\sigma_i} V \tag{A.19}$$

where we make the following additional substitution:

$$\alpha_i = c_i^{\sigma_i}$$

Plugging in equation (A.19) in the primal expression (equation (A.16)) we can derive an implicit function for the variable P^c:

$$\sum_i \frac{\alpha_i}{K\rho_i} \left(\frac{P_i}{A\lambda_i P^c} \right)^{1-\sigma_i} \equiv 1 \tag{A.20}$$

Equation (A.20) will be called the CRESH dual price expression that is clearly the counterpart to the CES dual price equation. It will be shown below that P^c is a price index, but that in only special cases will it equal the average price. This implies:

$$P.V = \sum_i P_i X_i \neq P^c.V$$

We can express the ratio of P to P^c as:

$$\frac{P}{P^c} = \frac{1}{P^c} \frac{\sum_i P_i X_i}{V} = \frac{1}{P^c} \frac{1}{V} \sum_i \left[P_i \alpha_i \left(A\lambda_i \right)^{\sigma_i - 1} \left(\frac{P^c}{P_i} \right)^{\sigma_i} V \right] = \sum_i \alpha_i \left(\frac{P_i}{A\lambda_i P^c} \right)^{1-\sigma_i}$$

The formula for shares can be expressed as:

$$s_i = \frac{P_i X_i}{P.V} = \frac{\alpha_i \left(\frac{P_i}{A\lambda_i P^c} \right)^{1-\sigma_i} P^c V}{P.V} = \frac{P^c}{P} \alpha_i \left(\frac{P_i}{A\lambda_i P^c} \right)^{1-\sigma_i} \tag{A.21}$$

And given the expression for the price ratio above, the final expression for shares is:

$$s_i = \frac{\alpha_i \left(\dfrac{P_i}{A\lambda_i P^c} \right)^{1-\sigma_i}}{\sum_k \alpha_k \left(\dfrac{P_k}{A\lambda_k P^c} \right)^{1-\sigma_k}} \tag{A.22}$$

We can recover an expression for the Lagrangian multiplier, Λ, starting with equation (A.16). Multiply each side by the ratio of X_i/ρ_i and take the sum over i. The right-hand side will evaluate to $K\Lambda$ and thus we have:

$$\Lambda = \sum_i \frac{P_i X_i}{K\rho_i} \tag{A.23}$$

In the case of a uniform σ, it can be shown that the price ratio, P/P^c is constant, i.e. the percentage change of each is identical and the ratio only depends on initialization. The expression for the ratio of the two prices in the case of uniformity is:

$$\frac{P}{P^c} = (P^c)^{\sigma-1} \sum_i \alpha_i \left(\frac{P_i}{A\lambda_i} \right)^{1-\sigma}$$

The dual price expression becomes:

$$\sum_i \alpha_i \left(\frac{P_i}{A\lambda_i} \right)^{1-\sigma} = K\rho \left(P^c\right)^{1-\sigma}$$

Substituting out the summation term, the resulting expression is:

$$\frac{P}{P^c} = (P^c)^{\sigma-1} K\rho \left(P^c\right)^{1-\sigma} = K\rho$$

In most many cases with default initializations, the product of $K\rho$ will equal 1. In this case, with a uniform substitution elasticity, there is a one-to-one correspondence between the CRESH (CES) price index, P^c, and the average price, P.

The own- and cross-price elasticities can be derived from equation (A.19):

$$\epsilon_{ij} = \frac{\partial X_i}{\partial P_j} \frac{P_j}{X_i} = \sigma_i \frac{\partial P^c}{\partial P_j} \frac{P_j}{P^c} - \sigma_i \delta_{ij} \tag{A.24}$$

One of the consistency requirements of the matrix of elasticities is that the inner product of each column with the column shares must equal 0:

$$\sum_i \epsilon_{ij} s_i = 0$$

where s_i represents the input shares. This implies the following:

$$\sum_i \epsilon_{ij} s_i = \sum_i s_i \left[\sigma_i \frac{\partial P^c}{\partial P_j} \frac{P_j}{P^c} - \sigma_i \delta_{ij} \right] = 0$$

The elasticity expression for the price index can then be isolated to yield:

$$\frac{\partial P^c}{\partial P_j} \frac{P_j}{P^c} = \frac{s_j \sigma_j}{\sum_i s_i \sigma_i}$$

The matrix of own- and cross-price elasticities is therefore:

$$\epsilon_{ij} = \sigma_i \left[\frac{s_j \sigma_j}{\sum_k s_k \sigma_k} - \delta_{ij} \right] \tag{A.25}$$

The ES (Allen-Uzawa pairwise partial ES) are related to the demand elasticities using the following expression:

$$\sigma^a_{ij} = \frac{\epsilon_{ij}}{s_j} = \frac{\sigma_i \sigma_j}{\sum_k s_k \sigma_k} - \frac{\sigma_i \delta_{ij}}{s_j} \tag{A.26}$$

Implementation in levels

Calibration is relatively straightforward. Given the σ parameters the ρ parameters can be calibrated. Equation (A.19) can be used to calibrate the CRESH dual share parameters, α_i, given initial values for A, λ_i, V, P_i, X_i and P^c. If prices and the technology variables are initialized at 1, the share parameters will reflect the initial value shares. The primal share parameters are calibrated using $c_i = \alpha_i^{1/\sigma_i}$. Finally, K can be calibrated using the CRESH primal function, equation (A.15).

Model implementation uses only two equations, equation (A.19) and (A.20) where the endogenous variables are X_i and P^c, given V, P_i and the parameters of the function.

Converting to percent differences

The CRESH demand function is not easy to interpret. In the ORANI/MONASH tradition, a linearized form can be derived that helps with the interpretation of the changes in input demand (X_i) and also eliminates Λ from the expression.

The linearized form of equation (A.19) is the following:

$$\dot{X}_i = \dot{V} + (\sigma_i - 1)\dot{A} + (\sigma_i - 1)\dot{\lambda}_i + \sigma_i \dot{P}^c - \sigma_i \dot{P}_i \tag{A.27}$$

Similarly, the linearized version of the dual price expression, equation (A.20) takes the following form:

$$\sum_j B_j \frac{\sigma_j - 1}{\rho_j} \left[\dot{P}_j - \dot{A} - \dot{\lambda}_j - \dot{P}^c\right] \equiv 0$$

where (after elimination of the parameter K):

$$B_j = \alpha_j \left(\frac{P_j}{A\lambda_j P^c}\right)^{1-\sigma_j}$$

Further, the B_j can be divided by the sum of the B_j and thus be transformed into the shares, s_j. As well, the expression $(\sigma_j - 1)/\rho_j$ can be replaced with σ_j. This leads to the following formula for the change in the CRESH price index:

$$\dot{P}^c = \frac{\sum_j s_j \sigma_j \left[\dot{P}_j - \dot{A} - \dot{\lambda} j\right]}{\sum_k s_j \sigma_j}$$

Define a modified cost share parameter using the following formula:

$$\varsigma_j = \frac{\sigma_j s_j}{\sum_k \sigma_k s_k} \tag{A.28}$$

Then the log-linearized expression for the CRESH price index simplifies to the following:

$$\dot{P}^c = \sum_j \varsigma_j \dot{P}_j - \sum_j \varsigma_j \dot{\lambda}_j - \dot{A} \tag{A.29}$$

Inserting equation (A.29) into equation (A.27) yields the final expression for the percent change in X_i as a function of the change in the aggregate volume, V, the component prices, P_i, and changes in technology or preferences, A and λ_i:

$$\dot{X}_i = \dot{V} - \dot{A} - \sigma_i \left[\dot{P}_i - \sum_j \varsigma_j \dot{P}_j\right] + \sigma_i \left[\dot{\lambda}_i - \sum_j \varsigma_j \dot{\lambda}_j\right] - \dot{\lambda}_i \tag{A.30}$$

This final expression is very similar to the linearized expression in Dixon and Rimmer (2002), equation (26.13) on page 221. The main difference is that the technology parameters, λ_i, are expressed as multiplicative factors of the inputs whereas in the Dixon and Rimmer volume they are expressed as inverses—thus the reversal in sign.[10] This requires some interpretation when undertaking technology shocks. For example, a 10 percent improvement in labor efficiency using the interpretation of this volume leads to the following expression:

$$\lambda_t = 1.1\lambda_{t-1}$$

[10] The expression also incorporates the aggregate technology shifter represented by the variable A.

Using the Dixon and Rimmer interpretation, the same shock is implemented as:

$$B_t = B_{t-1}(1 - 0.1/1.1) = 0.9091 B_{t-1}$$

Or more generally, a δ percent shock in this volume translates to a shock of $-\delta/(1+\delta)$ percent shock in Dixon and Rimmer. Similarly, if the Dixon and Rimmer shock is Δ, assumed to be positive, this translates into a shock of $\Delta/(1-\Delta)$. So a ten percent Dixon and Rimmer shock is equal to a 11.1 percent shock using the multiplicative form of technological change.

A.3.2 The CRETH function

The *constant ratios of elasticities of transformation, homethetic* (CRETH) function is an analogous generalization of the CET function that allows for pair-wise differences in the transformation elasticities. This section lists the main results without the same level of detail as for the CRESH functional form. The CRETH primal function starts with an implicit functional form:

$$F(X_i, V) = \sum_i \frac{b_i}{\nu_i} \left(\frac{A\lambda_i X_i}{V} \right)^{\nu_i} \equiv K \tag{A.31}$$

using the same notation as above where X_i are the supply components, λ_i are technology or preference shifters and V is total supply. In this formulation K is a calibrated parameter. We can see that if the exponents, ν_i, are identical across all i, the CRETH function becomes the CET function, where the primal parameters are given by $g_i = b_i/(K\nu)$.

The CRETH specification can be implemented with two sets of equations. The first determines the allocation of component i and is virtually identical to its CET counterpart with the caveat that the price index in the denominator, P^c, is not equal to the aggregate price, similar to the price index of the CRESH function. The second is the CRETH dual price expression that provides an implicit definition of the CRETH price index, P^c.

$$X_i = \gamma_i \left(A\lambda_i \right)^{-\omega_i - 1} \left(\frac{P_i}{P^c} \right)^{\omega_i} V \tag{A.32}$$

$$\sum_i \frac{\gamma_i}{K\nu_i} \left(\frac{P_i}{A\lambda_i P^c} \right)^{1+\omega_i} \equiv 1 \tag{A.33}$$

where we make the following additional substitutions:

$$\gamma_i = b_i^{-\omega_i}$$

$$\omega_i = \frac{1}{\nu_i - 1} \iff \nu_i = \frac{\omega_i + 1}{\omega_i}$$

Similarly in percent differences these equations become:

$$\dot{X}_i = \dot{V} - \dot{A} + \omega_i \left[\dot{P}_i - \sum_j \varsigma_j \dot{P}_j \right] - \omega_i \left[\dot{\lambda}_i - \sum_j \varsigma_j \dot{\lambda}_j \right] - \dot{\lambda}_i \tag{A.34}$$

$$\dot{P}^c = \sum_j \varsigma_j \dot{P}_j - \sum_j \varsigma_j \dot{\lambda}_j - \dot{A} \tag{A.35}$$

where the modified share parameter is given by:

$$\varsigma_j = \frac{\omega_j s_j}{\sum_k \omega_k s_k} \tag{A.36}$$

A.4 Modified CES and CET that incorporate additivity

The standard CET supply allocation specification does not preserve physical additivity, i.e. the sum of the volume components is not necessarily equal to the total volume. There are a number of alternative specifications that do preserve volume homogeneity, for example the multinomial logit. One alternative, described below, uses a modified form of the CET preference function. This specification has been used for labor and land supply allocations (see respectively Dixon and Rimmer (2006) and Giesecke et al. (2013)).

A.4.1 The CET implementation

The CET alternative involves solving the following optimization:

$$\max_{X_i} U = \left[\sum_i g_i \left(\lambda_i P_i X_i\right)^{\nu}\right]^{1/\nu}$$

subject to the constraint:

$$V = \sum_i X_i$$

The variable definitions are similar to above, X_i are the volume components, P_i are the relevant component prices and V is aggregate volume. The λ_i parameters are preference parameters. The CET utility function is not simply a function of the volumes, but explicitly a function of the preference-adjusted revenues of the individual components. The closed-form solution to the above system is the following set of equations:

$$X_i = \gamma_i V \left(\frac{\lambda_i P_i}{P^c}\right)^{\omega} \tag{A.37}$$

$$P^c = \left[\sum_i \gamma_i \left(\lambda_i P_i\right)^{\omega}\right]^{1/\omega} \tag{A.38}$$

Both equations are similar to their standard CET counterparts, but with some differences. First, P^c is a price index, but it is not the average price of the components, i.e. $P^c X \neq \sum_i P_i X_i$. Second, this price index is based on ω not $1+\omega$ as in the standard CET dual price expression. The revenue correct price index is defined by the following formula:

$$P = \frac{\sum_i \gamma_i \lambda_i^{\omega} P_i^{\omega+1}}{\sum_i \gamma_i \lambda_i^{\omega} P_i^{\omega}} = \frac{\sum_i \gamma_i \lambda_i^{\omega} P_i^{\omega+1}}{\left(P^c\right)^{\omega}} = \sum_i \gamma_i P_i \left(\frac{\lambda_i P_i}{P^c}\right)^{\omega} = \sum_i \frac{X_i}{V} P_i \tag{A.39}$$

The other transformations include:

$$\gamma_i = {g_i}^{1+\omega}$$

$$\omega = \frac{\nu}{1-\nu} \iff \nu = \frac{\omega}{1+\omega}$$

It is worth noting that the relation between ω and ν differs from the standard CET relation as the respective formula is inverted. The implication of this is that ν is bounded below by 0 instead of ∞, but is otherwise positive over the entire (positive) range of ω. And, in both the standard and adjusted CET ν converges to 1 as ω converges to ∞. As regards calibration, there is an extra degree of freedom as the value for utility is not specified. It is easiest to simply set P^c to 1 as for given P_i and λ_i the calibration of the γ parameters is straightforward:

$$\gamma_i = \frac{X_i}{V}\left(\frac{\lambda_i P_i}{P^c}\right)^{-\omega}$$

If prices and technology or preference parameters are initialized at 1, the calibrated γ parameters are equal to the initial volume shares.

Converting this to a Monash-style equation in percent differences, the derived supply function is:

$$\dot{X}_i = \dot{V} + \omega\left[\dot{P}_i + \dot{\lambda}_i - \sum_{j=1}^{n} \frac{X_j}{V}\left(\dot{P}_j + \dot{\lambda}_j\right)\right]$$

This equation uses volume shares as weights for cross-price (and cross-preference) effects. In the standard CET formulation, value shares are used as weights.

The standard specification needs some modifications for two special cases—perfect transformation and perfect immobility. The case of perfect transformation, i.e. a transformation elasticity of ∞, leads to all prices moving in unison with the aggregate price index. Thus equation (A.37) is replaced with the following expression:

$$\lambda_i P_i = P^c$$

where λ_i is calibrated to the initial price ratios. The price index expression, equation (A.38) is replaced with the volume constraint:

$$V = \sum_i X_i$$

In the model implementation of the adjusted CET, this latter expression can be used in all cases and can replace equation (A.38).

The case of zero mobility is readily implemented by dropping completely equation (A.38) (or its equivalent, i.e. the volume adding up constraint). With a transformation elasticity of 0, the price composite index in equation (A.37) simply drops out and the volume components are in strict proportion to the aggregate volume.

A.4.2 ACET and twists

It is relatively easy to show that the analogous twist formulas for the ACET are:

$$1 + \pi_{1,t} = \left[\frac{1 + tw_t}{1 + s_{1,t-1} tw_t}\right]^{1/\omega} \qquad 1 + \pi_{2,t} = \left[\frac{1}{1 + s_{1,t-1} tw_t}\right]^{1/\omega}$$

where the volume shares replace the value shares from the standard CET twists and the exponent differs.[11]

A.4.3 Generalizing to CRETH utility function

The allocation mechanism described above can be generalized to a CRETH utility function that allows for pair-wise transformation elasticities to differ. The optimization is setup according to the following:

$$\max U$$

subject to the constraints:

$$\sum_i \frac{c_i}{\nu_i} \left(\frac{\lambda_i P_i X_i}{U}\right)^{\nu_i} = K$$

$$V = \sum_i X_i$$

In the case of the CRETH function, utility is defined implicitly.

The first-order equations can be transformed to yield the following set of equations that are readily implemented:

$$X_i = \gamma_i \left(\sum_j \frac{X_j}{\nu_j}\right)^{(1+\omega_i)} \left(\frac{\lambda_i P_i}{U}\right)^{\omega_i} \tag{A.40}$$

$$V = \sum_i X_i \tag{A.41}$$

These two equations determine jointly X_i and U. We also have the following set of relations:

$$\gamma_i = \left(\frac{c_i}{K}\right)^{1+\omega_i}$$

$$\omega_i = \frac{\nu_i}{1 - \nu_i} \iff \nu_i = \frac{\omega_i}{1 + \omega_i} \iff 1 - \nu_i = \frac{1}{1 + \omega_i}$$

Calibration is relatively straightforward. For given initial values of P_i, X_i, λ_i and U, equation (A.40) can be used to calibrate the share parameters, γ_i. The c_i parameters can be recovered from the formula above linking the c and γ parameters given a value for K, which without loss of generality can be set to an initial value of 1.

Similar to the formulas linking the primal exponent and the transformation elasticity for the CET, the relations for the adjusted CRETH are different from the standard CRETH relations where the relation is $\omega_i = 1/(\nu_i - 1)$.

The formulas above can be transformed into equivalent equations that compare more readily to the CET-style equations. First we can define P^c using the following expression:

$$P^c = \frac{U}{\sum_j \frac{X_j}{\nu_j}}$$

[11] These formulas are consistent with no change in the ACET composite price. However, the average price may vary as well as utility.

and this latter expression for P^c can be substituted into equation (A.40). It can be shown subsequently that P^c can be defined implicitly by the equivalent of the CRETH dual price expression giving the following set of equations that determines X_i and P^c given only P_i (and the functional parameters):

$$X_i = \gamma_i \left(\sum_j \frac{X_j}{\nu_j} \right) \left(\frac{\lambda_i P_i}{P^c} \right)^{\omega_i} \tag{A.42}$$

$$\sum_i \frac{\gamma_i}{\nu_i} \left(\frac{\lambda_i P_i}{P^c} \right)^{\omega_i} = 1 \tag{A.43}$$

The price index given by P^c is defined implicitly in equation (A.43), and like in the case of the CET, it is not necessarily equal to the average price of the aggregate volume. The other difference would be in the calibration of the γ parameters. In the equations above, the γ parameters are adjusted by ν_i, whereas in the case of the CET specification, the γ parameters integrate the (common) substitution parameter.

It turns out that the set of equations above, equations (A.42) and (A.43) are linearly dependent. For implementation purposes the model should implement equations (A.42) and (A.41).

In log-linear difference form, equation (A.42) becomes the following:

$$\dot{X}_i = \sum_j \xi_j \dot{X}_j + \omega_i \left[\dot{P}_i - \sum_j \varsigma_j \dot{P}_j \right] + \omega_i \left[\dot{\lambda}_i - \sum_j \varsigma_j \dot{\lambda}_j \right] \tag{A.44}$$

where the weights are defined by the following formulas:

$$\xi_j = \frac{X_j / \nu_j}{\sum_k X_k / \nu_k}$$

$$\varsigma_j = \frac{(1+\omega_j) X_j}{\sum_k (1+\omega_k) X_k}$$

Both sets of weights simplify to volume weights when the elasticities are uniform as in the case of the CET. The first term on the right hand side aggregates two effects. The first effect is the aggregate volume increase, i.e. $\dot{V}$. The second effect is a volume re-allocation effect. The expression can be converted to the following:

$$\dot{X}_i = \dot{V} + \sum_j \varphi_j \dot{X}_j + \omega_i \left[\dot{P}_i - \sum_j \varsigma_j \dot{P}_j \right] + \omega_i \left[\dot{\lambda}_i - \sum_j \varsigma_j \dot{\lambda}_j \right] \tag{A.45}$$

where φ is defined by:

$$\varphi_j = \xi_j - \frac{X_j}{V}$$

From this formula it is clear that the re-allocation effect is 0 in the case of the CET as the ξ weights are equated to the volume shares.

A.4.4 The CES implementation

The adjusted CET (and CRETH) functions replace their counterparts for the allocation problem that preserves additivity. Analogous specifications exist for the CES and CRESH functions that emulate the implementation of their standard counterparts but also allow for additivity.

The CES alternative involves solving the following optimization:

$$\min_{X_i} U = \left[\sum_i a_i \left(\lambda_i P_i X_i \right)^{\rho} \right]^{1/\rho}$$

subject to the constraint:

$$V = \sum_i X_i$$

As in the case of the adjusted CET, the adjusted CES utility function is a function of the preference adjusted cost components. The closed-form solution to the above system is the following set of equations:

$$X_i = \alpha_i V \left(\frac{P^c}{\lambda_i P_i} \right)^{\sigma} \tag{A.46}$$

$$P^c = \left[\sum_i \alpha_i \left(\lambda_i P_i\right)^{-\sigma}\right]^{-1/\sigma} \tag{A.47}$$

Both equations are similar to their standard CES counterparts, but with some differences. First, P^c is a price index, but it is not the average price of the components, i.e. $P^c X \neq \sum_i P_i X_i$. Second, this price index is based on $-\sigma$ not $1-\sigma$ as in the standard CES dual price expression. The revenue correct price index is defined by the following formula:

$$P = \frac{\sum\limits_i \alpha_i \lambda_i^\sigma P_i^{1-\sigma}}{\sum\limits_i \alpha_i \lambda_i^\sigma P_i^\sigma} = \frac{\sum\limits_i \alpha_i \lambda_i^\sigma P_i^{1-\sigma}}{\left(P^c\right)^{-\sigma}} = \sum_i \alpha_i P_i \left(\frac{\lambda_i P_i}{P^c}\right)^{-\sigma} = \sum_i \frac{X_i}{V} P_i \tag{A.48}$$

The other transformations include:

$$\alpha_i = {a_i}^{1-\sigma}$$

$$\sigma = \frac{\rho}{\rho - 1} \iff \rho = \frac{\sigma}{\sigma - 1}$$

It is worth noting that the relation between σ and ρ differs from the standard CES relation as the respective formula is inverted. The implication of this is that ρ is bounded below by 0 instead of $-\infty$. It decreases towards $-\infty$, as σ increases towards 1, which is a discontinuity point. It decreases from ∞ towards 1 as σ increases from 1 to ∞.

It is relatively easy to show that the following simple expression holds for the utility function:

$$U = P^c V \tag{A.49}$$

As regards calibration, there is an extra degree of freedom as the value for utility is not specified. It is easiest to simply set P^c to 1 as for given P_i and λ_i the calibration of the α parameters is straightforward:

$$\alpha_i = \frac{X_i}{V}\left(\frac{\lambda_i P_i}{P^c}\right)^\sigma$$

If prices and technology or preference parameters are initialized at 1, the calibrated α parameters are equal to the initial volume shares.

Converting this to a Monash-style equation in percent differences, the derived demand function is:

$$\dot{X}_i = \dot{V} - \sigma\left[\dot{P}_i + \dot{\lambda}_i - \sum_{j=1}^{n} \frac{X_j}{V}\left(\dot{P}_j + \dot{\lambda}_j\right)\right]$$

This equation uses volume shares as weights for cross-price (and cross-preference) effects. In the standard CES formulation, value shares are used as weights.

The normalized ACES

Similar to the standard CES formulation, one can normalize the variables so that they are all 1 at some reference point. The normalized expressions are the following:

$$X_i = V\left(\frac{P^c}{\lambda_i P_i}\right)^\sigma$$

and

$$P^c = \left[\sum_i \alpha_i \left(\lambda_i P_i\right)^{-\sigma}\right]^{-1/\sigma}$$

where the share parameters on the composite price index are the initial *volume* shares, i.e. $\alpha_i = r_{i,0} = X_{i,0}/V_0$.

A.4.5 Using twists with the adjusted CES

The 'twist' idea described for the normal CES can be applied to the adjusted CES. The concept is somewhat different given the type of optimization problem posed. Rather than change the share parameters in a given direction with cost neutrality, the idea is to change the share parameters with utility neutrality. The problem posed, therefore, is to change the ratio of demand for two goods by a specified amount, while maintaining the same level of utility.

The ratio of the two components is given by the following expression using equation (A.46) as the starting point:

$$R = \frac{\alpha_1 \lambda_2 {P_2}^\sigma}{\alpha_2 \lambda_1 {P_1}^\sigma}$$

The idea is to move the initial ratio, R_{t-1} to R_t by tw percent.

$$\frac{R_t}{R_{t-1}} = (1 + tw_t)$$

while holding U constant. The two expressions above imply that the preference shifters, given by the π parameters, are linked via the following expression:

$$1 + \pi_2 = (1 + \pi_1)(1 + tw)^{1/\sigma} \tag{A.50}$$

Given equation (A.49), holding U constant is equivalent to holding the price index, P^c, constant as well (for a fixed aggregate volume). Thus we can solve the following equation for the parameter π_1:

$$\begin{aligned}
(P^c_{t-1})^{-\sigma} &= \alpha_1 (P_{1,t-1}\lambda_{1,t-1})^{-\sigma} + \alpha_2 (P_{2,t-1}\lambda_{2,t-1})^{-\sigma} \\
&= \alpha_1 (P_{1,t-1}\lambda_{1,t-1}(1 + \pi_{1,t}))^{-\sigma} + \alpha_2 (P_{2,t-1}\lambda_{2,t-1}(1 + \pi_{2,t}))^{-\sigma} \\
&= \alpha_1 (P_{1,t-1}\lambda_{1,t-1}(1 + \pi_{1,t}))^{-\sigma} + \alpha_2 \left(P_{2,t-1}\lambda_{2,t-1}(1 + \pi_{1,t})(1 + tw)^{1/\sigma}\right)^{-\sigma} \\
&= (P^c_t)^{-\sigma}
\end{aligned}$$

The π variables represent the growth (either positive or negative) that will be applied to the preference parameters under the assumption of utility-preserving preference shifts. This formula can be written in terms of the initial volume shares, $r_i = X_i/V$, simplified and re-arranged to yield:

$$(1 + \pi_1)^\sigma = r_1 + \frac{r_2}{1 + tw}$$

and when re-inserted in equation (A.50) we get:

$$(1 + \pi_2)^\sigma = \left(r_1 + \frac{r_2}{1 + tw}\right)(1 + tw)$$

The final formulas for the two twist parameters only depend on the initial volume shares, the substitution elasticity and the level of the 'twist':

$$\pi_1 == \left[\frac{1 + r_1\tau}{1 + \tau}\right]^{1/\sigma} - 1 \tag{A.51}$$

$$\pi_2 = [1 + r_1\tau]^{1/\sigma} - 1 \tag{A.52}$$

It is possible to generalize these formulas by partitioning the set of CES components into two sets—a set indexed by 1 that is the target set, and a set indexed by 2 that is the complement. For example, think of a set of electricity technologies that includes conventional and advanced. It is possible then to provide the same twist to all of the new technologies relative to the conventional technologies. The only change in the formulas above is that the volume share variable for the single component is replaced by the sum of the volume shares for the bundle of components:

$$\lambda_{1,t} = (1 + \pi_{1,t})\lambda_{1,t-1} = \left[\sum_{i \in 1} r_{i,t-1} + \frac{\sum_{i \in 2} r_{i,t-1}}{1 + tw_t}\right]^{1/\sigma} \lambda_{1,t-1}$$

$$\lambda_{2,t} = (1 + \pi_{2,t})\lambda_{2,t-1} = \left[\sum_{i \in 1} (1 + tw_t) r_{i,t-1} + \sum_{i \in 2} r_{i,t-1}\right]^{1/\sigma} \lambda_{2,t-1}$$

Converting to percent differences

The π factors reflect a percentage change in the relevant productivity factors for each of the components. Using a Taylor series approximation, the formulas above can be converted to a linear equation that is used by the Monash-style models. For the first component, we have:

$$\pi_1 = F(tw) = \left[r_1 + \frac{r_2}{1 + tw}\right]^{1/\sigma} - 1 \approx F(0) + tw.F'(0) = -tw\frac{r_2}{\sigma}$$

For the second component we have:

$$\pi_2 = F(tw) = [r_1(1 + tw) + r_2]^{1/\sigma} - 1 \approx F(0) + tw.F'(0) = tw\frac{r_1}{\sigma}$$

A.4.6 Generalizing to CRESH utility function

The CES utility function described above can be generalized to a CRESH-style utility function that allows for pair-wise substitution elasticities to differ. The optimization is setup according to the following:

$$\max U$$

subject to the constraints:

$$\sum_i \frac{c_i}{\rho_i}\left(\frac{\lambda_i P_i X_i}{U}\right)^{\rho_i} = K$$

$$V = \sum_i X_i$$

In the case of the CRESH function, utility is defined implicitly.

The first-order equations can be transformed to yield the following set of equations that are readily implemented:

$$X_i = \alpha_i \left(\sum_j \frac{X_j}{\rho_j}\right)^{(1-\sigma_i)} \left(\frac{\lambda_i P_i}{U}\right)^{-\sigma_i} \tag{A.53}$$

$$V = \sum_i X_i \tag{A.54}$$

These two equations determine jointly X_i and U. We also have the following set of relations:

$$\alpha_i = \left(\frac{c_i}{K}\right)^{1-\sigma_i}$$

$$\sigma_i = \frac{\rho_i}{\rho_i - 1} \iff \rho_i = \frac{\sigma_i}{\sigma_i - 1}$$

As in the case of the CRETH, we can define a CRESH price index, P^c, to get the final function forms:

$$X_i = \alpha_i \left(\sum_j \frac{X_j}{\rho_j}\right) \left(\frac{P^c}{\lambda_i P_i}\right)^{\sigma_i} \tag{A.55}$$

$$\sum_i \frac{\alpha_i}{\rho_i}\left(\frac{\lambda_i P_i}{P^c}\right)^{-\sigma_i} = 1 \tag{A.56}$$

It turns out that the set of equations above, equations (A.55) and (A.56) are linearly dependent. For implementation purposes the model should implement equations (A.55) and (A.54).

In log-linear difference form, equation (A.55) becomes the following:

$$\dot{X}_i = \sum_j \xi_j \dot{X}_j - \sigma_i\left[\dot{P}_i - \sum_j \varsigma_j \dot{P}_j\right] - \sigma_i\left[\dot{\lambda}_i - \sum_j \varsigma_j \dot{\lambda}_j\right] \tag{A.57}$$

where the weights are defined by the following formulas:

$$\xi_j = \frac{X_j/\rho_j}{\sum_k X_k/\rho_k}$$

$$\varsigma_j = \frac{(1-\sigma_j)X_j}{\sum_k (1-\sigma_k)X_k}$$

Both sets of weights simplify to volume weights when the elasticities are uniform as in the case of the CES. The first term on the right hand side aggregates two effects. The first effect is the aggregate volume increase, i.e. $\dot{V}$. The second effect is a volume re-allocation effect. The expression can be converted to the following:

$$\dot{X}_i = \dot{V} + \sum_j \varphi_j \dot{X}_j - \sigma_i\left[\dot{P}_i - \sum_j \varsigma_j \dot{P}_j\right] - \sigma_i\left[\dot{\lambda}_i - \sum_j \varsigma_j \dot{\lambda}_j\right] \tag{A.58}$$

where φ is defined by:

$$\varphi_j = \xi_j - \frac{X_j}{V}$$

From this formula it is clear that the re-allocation effect is 0 in the case of the CES as the ξ weights are equated to the volume shares.

A.5 Summary

Figures A.1 and A.2 depict respectively the relation between the CES substitution elasticity, σ and its primal counterpart ρ and the relation between the CET transformation elasticity, ω, and its primal counterpart, ν. The figures clearly show the sharp departures of the respective curves for the standard version and the version that preserves volume additivity for low values of the respective elasticities. The two curves eventually converge for higher elasticity values.

Table A.6 summarizes the key formulas and relations for all of the functional forms described above.

Figure A.1: **The CES exponent (ρ) as a function of the substitution elasticity**

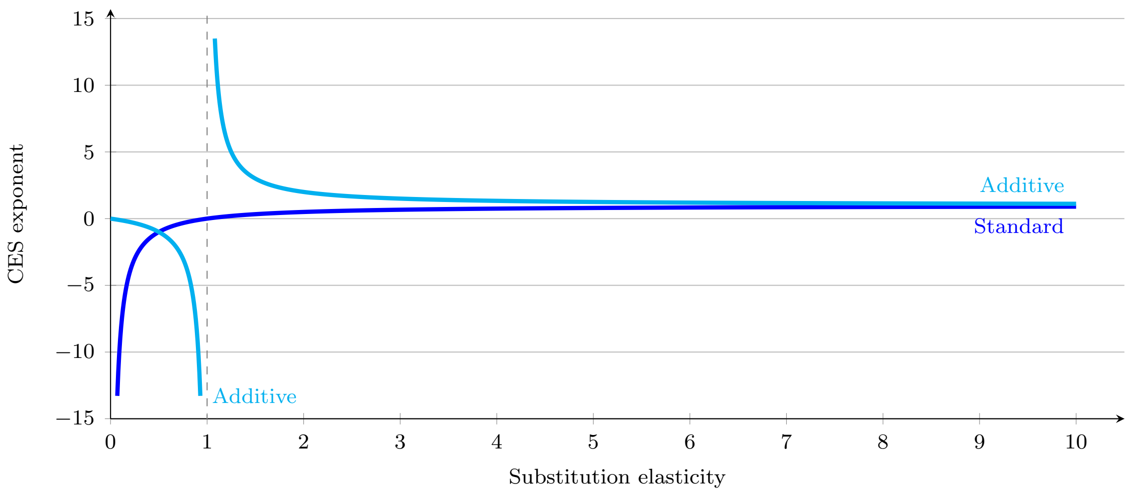

Figure A.2: **The CET exponent (ν) as a function of the transformation elasticity**

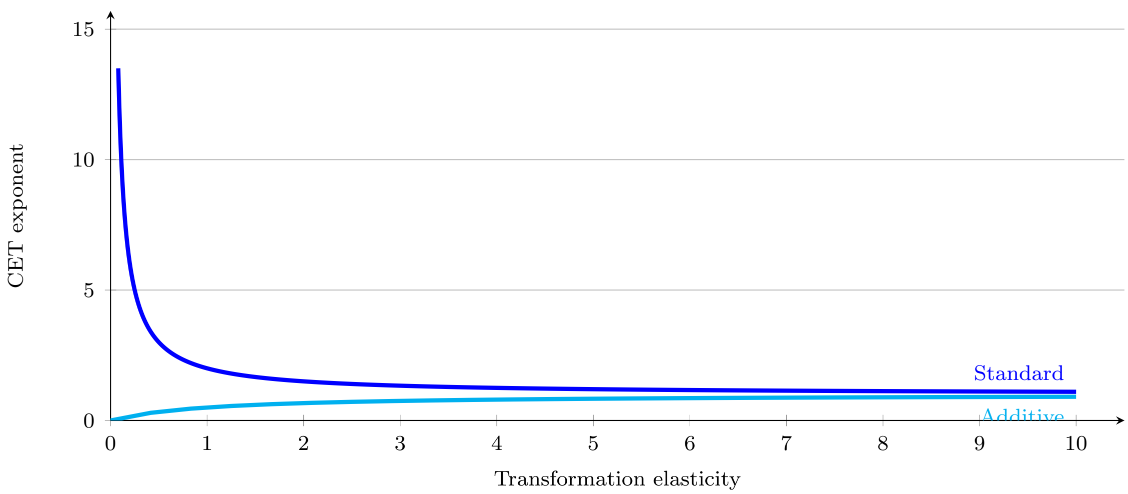

Table A.6: Summary of formulas for the CES/CRESH and CET/CRETH functions

CES		*CET*	
$X_i = \alpha_i (A\lambda_i)^{\sigma-1} V \left(\frac{P}{P_i}\right)^{\sigma}$		$X_i = \gamma_i (A\lambda_i)^{-\omega-1} V \left(\frac{P_i}{P}\right)^{\omega}$	
$P = \frac{1}{A}\left[\sum_i \alpha_i \left(\frac{P_i}{\lambda_i}\right)^{1-\sigma}\right]^{1/(1-\sigma)}$		$P = \frac{1}{A}\left[\sum_i \gamma_i \left(\frac{P_i}{\lambda_i}\right)^{1+\omega}\right]^{1/(1+\omega)}$	
$\sigma = \frac{1}{1-\rho} \geq 0$		$\omega = \frac{1}{\nu-1} \geq 0$	
$\rho = \frac{\sigma-1}{\sigma}$		$\nu = \frac{\omega+1}{\omega}$	
$\sigma = 0 \Rightarrow \rho = -\infty$	Leontief	$\omega = 0 \Rightarrow \nu = \infty$	Leontief
$0 < \sigma < 1 \Rightarrow \rho < 0$			
$\sigma = 1 \Rightarrow \rho = 0$	Cobb-Douglas	$0 < \omega < \infty \Rightarrow \nu > 0$	
$1 < \sigma < \infty \Rightarrow \rho > 0$			
$\sigma = \infty \Rightarrow \rho = 1$	Perfect substitution	$\omega = \infty \Rightarrow \nu = 1$	Perfect transformation
CRESH		*CRETH*	
$X_i = \alpha_i \, (A\lambda_i)^{\sigma_i-1} \, V \left(\frac{P}{P_i}\right)^{\sigma_i}$		$X_i = \gamma_i \, (A\lambda_i)^{-\omega-1} \, V \left(\frac{P_i}{P}\right)^{\omega_i}$	
$1 = \sum_i \frac{\alpha_i}{K\rho_i} \left(\frac{P_i}{A\lambda_i P}\right)^{1-\sigma_i}$		$1 = \sum_i \frac{\gamma_i}{K\nu_i} \left(\frac{P_i}{A\lambda_i P}\right)^{1+\omega_i}$	
$\sigma_i = \frac{1}{1-\rho_i} \geq 0$		$\omega_i = \frac{1}{\nu_i-1} \geq 0$	
$\rho_i = \frac{\sigma_i-1}{\sigma_i}$		$\nu_i = \frac{\omega_i+1}{\omega_i}$	
Additive CES		*Additive CET*	
$X_i = \alpha_i V \left(\frac{P}{\lambda_i P_i}\right)^{\sigma}$		$X_i = \gamma_i V \left(\frac{\lambda_i P_i}{P}\right)^{\omega}$	
$P = \left[\sum_i \alpha_i \, (\lambda_i P_i)^{-\sigma}\right]^{-1/\sigma}$		$P = \left[\sum_i \gamma_i \, (\lambda_i P_i)^{\omega}\right]^{1/\omega}$	
$\sigma = \frac{\rho}{\rho-1} \geq 0$		$\omega = \frac{\nu}{1-\nu} \geq 0$	
$\rho = \frac{\sigma}{\sigma-1}$		$\nu = \frac{\omega}{\omega+1}$	
$\sigma = 0 \Rightarrow \rho = 0$	Leontief	$\omega = 0 \Rightarrow \nu = 0$	Leontief
$0 < \sigma < 1 \Rightarrow \rho < 0$			
$\sigma = 1-\epsilon \Rightarrow \rho = -\infty$			
$\sigma = 1+\epsilon \Rightarrow \rho = \infty$	Discontinuity	$0 < \omega < \infty \Rightarrow \nu > 0$	
$1 < \sigma < \infty \Rightarrow \rho > 0$			
$\sigma = \infty \Rightarrow \rho = 1$		$\omega = \infty \Rightarrow \nu = 1$	
Additive CRESH		*Additive CRETH*	
$X_i = \alpha_i \sum_j \frac{X_j}{\rho_j} \left(\frac{P}{\lambda_i P_i}\right)^{\sigma_i}$		$X_i = \gamma_i \sum_j \frac{X_j}{\nu_j} \left(\frac{\lambda_i P_i}{P}\right)^{\omega_i}$	
$1 = \sum_i \frac{\alpha_i}{\rho_i} \left(\frac{\lambda_i P_i}{P}\right)^{-\sigma_i}$		$1 = \sum_i \frac{\gamma_i}{\nu_i} \left(\frac{\lambda_i P_i}{P}\right)^{\omega_i}$	
$\sigma_i = \frac{\rho_i}{\rho_i-1} \geq 0$		$\omega_i = \frac{\nu_i}{1-\nu_i} \geq 0$	
$\rho_i = \frac{\sigma_i}{\sigma_i-1}$		$\nu_i = \frac{\omega_i}{\omega_i+1}$	

Appendix B

The CDE demand system

B.1 The CDE demand system

The *Constant Difference of Elasticities* (CDE) function is a generalization of the CES function, but it allows for more flexibility in terms of substitution effects across goods and for non-homotheticity.[1] The starting point is an implicitly additive indirect utility function (see Hanoch (1975)) from which we can derive demand using Roy's identity (and the implicit function theorem).

B.1.1 General form

A dual approach is used to determine the properties of the CDE function. The indirect utility function is defined implicitly via the following expression:

$$V(p,u,Y) = \sum_{i=1}^{n} \alpha_i u^{e_i b_i} \left(\frac{p_i}{y}\right)^{b_i} \equiv 1 \tag{B.1}$$

where p is the vector of commodity prices, u is (per capita) utility and y is per capita income. Using Roy's identity and the implicit function theorem[2] we can derive demand, x, where v is the indirect utility function (defined implicitly):

$$x_i = -\frac{\partial v}{\partial p_i} / \frac{\partial v}{\partial Y} = -\left(\frac{\partial V}{\partial p_i} / \frac{\partial V}{\partial u}\right) / \left(\frac{\partial V}{\partial Y} / \frac{\partial V}{\partial u}\right) = -\left(\frac{\partial V}{\partial p_i} / \frac{\partial V}{\partial Y}\right) \tag{B.2}$$

This then leads to the following demand function:

$$x_i = \frac{\alpha_i b_i u^{e_i b_i} \left(\frac{p_i}{y}\right)^{b_i - 1}}{\sum\limits_j \alpha_j b_j u^{e_j b_j} \left(\frac{p_j}{y}\right)^{b_j}} \tag{B.3}$$

Implementation is easier if we define the following variable:

$$ZC_i = \alpha_i b_i u^{e_i b_i} \left(\frac{p_i}{y}\right)^{b_i} \tag{B.4}$$

Then the budget shares can be expressed as:

$$s_i = \frac{ZC_i}{\sum_j ZC_j} \tag{B.5}$$

and the demand expression is:

$$x_i = \frac{s_i}{p_i} y \tag{B.6}$$

[1] More detailed descriptions of the CDE can be found in Hertel et al. (1991), Surry (1993) and Hertel (1997).

[2] See Varian (1992), p. 109.

Implementation also requires evaluating u. This can be done by implementing equation (B.1) and inserting the expression for ZC:

$$\sum_{i=1}^{n} \frac{ZC_i}{b_i} \equiv 1 \tag{B.7}$$

B.1.2 Elasticities

In order to calibrate the CDE system, it is necessary to derive the demand and income elasticities of the CDE. The algebra is tedious, but straightforward.

The own-price elasticity is given by the following:

$$\varepsilon_i = \frac{\partial x_i}{\partial p_i}\frac{p_i}{x_i} = \frac{s_i\left[\sum_j s_j e_j b_j - e_i b_i\right]}{\sum_j s_j e_j} + b_i\left(1 - s_i\right) - 1 \tag{B.8}$$

In deriving the elasticity, we make use of the following formula that defines the elasticity of utility with respect to price (and again makes use of the implicit function theorem):

$$\frac{\partial u}{\partial p_i}\frac{p_i}{u} = -\frac{p_i}{u}\left(\frac{\partial V}{\partial p_i}\right) / \left(\frac{\partial V}{\partial u}\right) = -\frac{s_i}{\sum_j s_j e_j} \tag{B.9}$$

The price elasticity of utility is approximately the value share of the respective demand component as long as the weighted sum of the expansion parameters, e, is close to unity. The value (or budget) share is defined in the next equation:

$$s_i = \frac{p_i x_i}{y} \tag{B.10}$$

Letting $\sigma_i = 1 - b_i$ (or $b_i = 1 - \sigma_i$), we can also write:

$$\varepsilon_i = s_i\left[\sigma_i - \frac{e_i(1-\sigma_i)}{\sum_j s_j e_j} - \frac{\sum_j s_j e_j \sigma_j}{\sum_j s_j e_j}\right] - \sigma_i \tag{B.11}$$

With σ uniform, we also have:

$$\varepsilon_i = -\frac{s_i e_i (1-\sigma)}{\sum_j s_j e_j} - \sigma \tag{B.12}$$

With both e and σ uniform, the formula simplifies to:

$$\varepsilon_i = -s_i(1-\sigma) - \sigma = \sigma(s_i - 1) - s_i \tag{B.13}$$

Equation (B.13) reflects the own-price elasticity for the standard CES utility function. Finally, with e uniform but not σ, we have:

$$\varepsilon_i = s_i\left[2\sigma_i - 1 - \sum_j s_j \sigma_j\right] - \sigma_i \tag{B.14}$$

The derivation of the cross elasticities is almost identical and will not be carried out here. Combining both the own-and cross price elasticities, the matrix of substitution elasticities takes the following form where we use the Kronecker product, δ:[3]

$$\varepsilon_{ij} = s_j\left[-b_j - \frac{e_i b_i}{\sum_k s_k e_k} + \frac{\sum_k s_k e_k b_k}{\sum_k s_k e_k}\right] + \delta_{ij}(b_i - 1) \tag{B.15}$$

3 δ takes the value of 1 along the diagonal (i.e. when $i = j$) and the value 0 off-diagonal (i.e. when $i \neq j$).

Again, we replace b by $1-\sigma$, to get:

$$\varepsilon_{ij} = s_j \left[\sigma_j - \frac{e_i(1-\sigma_i)}{\sum\limits_k s_k e_k} - \frac{\sum\limits_k s_k e_k \sigma_k}{\sum\limits_k s_k e_k} \right] - \delta_{ij}\sigma_i \tag{B.16}$$

For uniform σ, equation (B.16) takes the form:

$$\varepsilon_{ij} = -\frac{e_i s_j (1-\sigma)}{\sum\limits_k s_k e_k} - \delta_{ij}\sigma \tag{B.17}$$

And with a uniform σ and e, i.e. the CES assumption, we have:

$$\varepsilon_{ij} = -s_j(1-\sigma) - \delta_{ij}\sigma = \sigma(s_j - \delta_{ij}) - s_j \tag{B.18}$$

Finally, for a uniform e only, the matrix of elasticities is:

$$\varepsilon_{ij} = s_j \left[\sigma_j - (1-\sigma_i) - \sum_k s_k \sigma_k \right] - \delta_{ij}\sigma_i \tag{B.19}$$

The income elasticities are derived in a similar fashion:

$$\eta_i = \frac{\partial x_i}{\partial Y}\frac{Y}{x_i} = \frac{1}{\sum\limits_k s_k e_k} \left[e_i b_i - \sum_k s_k e_k b_k \right] - (b_i - 1) + \sum_k b_k s_k \tag{B.20}$$

For this, we need the elasticity of utility with respect to income:

$$\frac{\partial u}{\partial Y}\frac{Y}{u} = -\frac{Y}{u}\left(\frac{\partial V}{\partial Y}\right) / \left(\frac{\partial V}{\partial u}\right) = \frac{1}{\sum\limits_k s_k e_k} \tag{B.21}$$

Note that for a uniform and unitary e, the income elasticity of utility is 1.
Replacing b with $1-\sigma$, equation (B.20) can be re-written to be:

$$\eta_i = \frac{1}{\sum\limits_k s_k e_k} \left[e_i(1-\sigma_i) + \sum_k s_k e_k \sigma_k \right] + \sigma_i - \sum_k s_k \sigma_k \tag{B.22}$$

With a uniform σ, the income elasticity becomes:

$$\eta_i = \frac{1}{\sum\limits_k s_k e_k} \left[e_i(1-\sigma) + \sigma \sum_k s_k e_k \right] = \frac{e_i(1-\sigma)}{\sum\limits_k s_k e_k} + \sigma \tag{B.23}$$

With e uniform, the income elasticity is unitary, irrespective of the values of the σ parameters.

From the Slutsky equation, we can calculate the compensated demand elasticities:

$$\xi_{ij} = \varepsilon_{ij} + s_j \eta_i = -\delta_{ij}\sigma_i + s_j \left[\sigma_j + \sigma_i - \sum_k s_k \sigma_k \right] \tag{B.24}$$

The cross-Allen partial elasticities are equal to the compensated demand elasticities divided by the share:

$$\sigma_{ij}^a = \sigma_j + \sigma_i - \sum_k s_k \sigma_k - \delta_{ij}\sigma_i / s_j \tag{B.25}$$

It can be readily seen that the difference of the partial elasticities is constant, hence the name of *constant difference in elasticities.*

$$\sigma_{ij}^a - \sigma_{il}^a = \sigma_j - \sigma_l \tag{B.26}$$

With a uniform σ, we revert back to the standard CES where there is equivalence between the CES substitution elasticity and the cross-Allen partial elasticity:

$$\sigma_{ij}^a = \sigma \tag{B.27}$$

B.1.3 Calibration of the CDE

Calibration assumes that we know the budget shares, the own uncompensated demand elasticities and the income elasticities. The weighted sum of the income elasticities must equal 1, so the first step in the calibration procedure is to make sure Engel's law holds. One alternative is to fix some (or none) of the income elasticities and re-scale the others using least squares. The problem is to minimize the following objective function:

$$\sum_{i\in\Omega}\left(\eta_i-\eta_i^0\right)^2$$

subject to

$$\sum_{i\in\Omega}s_i\eta_i=1-\sum_{i\notin\Omega}s_i\eta_i$$

where the set Ω contains all sectors where the income elasticity is not fixed, i.e. its complement contains those sectors with fixed income elasticities. The solution is:

$$\eta_i=\eta_i^0+s_i\frac{1-\sum\limits_{j\notin\Omega}s_j\eta_j-\sum\limits_{j\in\Omega}s_j\eta_j^0}{\sum\limits_{j\in\Omega}s_j^2}\qquad\forall i\in\Omega$$

Calibration of the σ parameters is straightforward given the own elasticities and the input value shares. The first step is to calculate the Allen partial elasticities, these are simply the income elasticity adjusted by the own elasticities divided by the budget shares:

$$\sigma_{ii}^a=\eta_i+\frac{\varepsilon_{ii}}{s_i}\tag{B.28}$$

Next, equation (B.25) is setup in matrix form:

$$\sigma_{ii}^a=A\sigma_i\tag{B.29}$$

where the matrix A has the form:

$$A=\begin{bmatrix}2-\dfrac{1}{s_1}-s_1 & -s_2 & \dots & -s_n\\ -s_1 & 2-\dfrac{1}{s_2}-s_2 & \dots & -s_n\\ \vdots & \vdots & \ddots & \vdots\\ -s_1 & -s_2 & \dots & 2-\dfrac{1}{s_n}-s_n\end{bmatrix}\tag{B.30}$$

or each element of A has the following formula:

$$a_{ij}=\delta_{ij}(2-1/s_i)-s_j$$

We can then solve for σ (and back-out the b parameters):

$$\sigma_i=A^{-1}\sigma_{ii}^a\tag{B.31}$$

There is nothing which guarantees the consistency of the calibrated σ parameters, which are meant to be positive. The calculation of the σ parameters depends only on the budget shares and the own-price uncompensated elasticities. If the calibrated σ parameters are not all positive, one could modify the elasticities until consistency is achieved. In practice, problems have occurred when a sector's budget share dominates total expenditure.

The e parameters are derived from Equation (B.22) and normalizing them so that their share weighted sum is equal to 1. Equation (B.22) can then be converted to matrix form and inverted:

$$B=\begin{bmatrix}s_1\sigma_1+(1-\sigma_1) & s_2\sigma_2 & \dots & s_n\sigma_n\\ s_1\sigma_1 & s_2\sigma_2+(1-\sigma_2) & \dots & s_n\sigma_n\\ \vdots & \vdots & \ddots & \vdots\\ s_1\sigma_1 & s_2\sigma_2 & \dots & s_n\sigma_n+(1-\sigma_n)\end{bmatrix}\tag{B.32}$$

or

$$b_{ij}=s_j\sigma_j+\delta_{ij}(1-\sigma_i)$$

Then the e parameters are derived from matrix inversion:

$$e_i = B^{-1}C_i = B^{-1}\left(\eta_i - \sigma_i + \sum_k s_k\sigma_k\right) \tag{B.33}$$

Calibration of the α parameters is based on equations (B.1) and (B.3). Start first with equation (B.3) and write it in terms relative to consumption of good 1, i.e.:

$$\frac{x_i}{x_1} = \frac{\alpha_i b_i u^{e_i b_i}\left(\dfrac{p_i}{Y}\right)^{b_i-1}}{\alpha_1 b_1 u^{e_1 b_1}\left(\dfrac{p_1}{Y}\right)^{b_1-1}} \tag{B.34}$$

This equation can be used to isolate α_i:

$$\alpha_i = \frac{x_i}{x_1}\frac{\alpha_1 b_1 u^{e_1 b_1}\left(\dfrac{p_1}{Y}\right)^{b_1-1}}{b_i u^{e_i b_i}\left(\dfrac{p_i}{Y}\right)^{b_i-1}} \tag{B.35}$$

and then inserted into equation (B.3):

$$\sum_{i=1}^{n}\alpha_i u^{e_i b_i}\left(\frac{p_i}{Y}\right)^{b_i} = \alpha_1 u^{e_1 b_1}\frac{b_1}{s_1}\left(\frac{p_1}{Y}\right)^{b_1}\left[\sum_{i=1}^{n}\frac{s_i}{b_i}\right] \equiv 1 \tag{B.36}$$

The final expression in equation (B.36) can be used to solve for α_1 since the formula must equal 1 by definition:

$$\alpha_1 = u^{-e_1 b_1}\frac{s_1}{b_1}\left(\frac{Y}{p_1}\right)^{b_1}\left[\sum_{i=1}^{n}\frac{s_i}{b_i}\right]^{-1} \tag{B.37}$$

Substituting back into equation (B.36) we get:

$$\alpha_i = \frac{x_i}{b_i}u^{-e_i b_i}\left(\frac{Y}{p_i}\right)^{b_i-1}\left[\sum_{j=1}^{n}\frac{s_j}{b_j}\right]^{-1} \tag{B.38}$$

The final calibration expression is then the following:

$$\alpha_i = \frac{s_i}{b_i}\left(\frac{Y}{p_i}\right)^{b_i}\frac{u^{-e_i b_i}}{\sum\limits_{j=1}^{n}\frac{s_j}{b_j}} \tag{B.39}$$

Utility is undefined in the base data and it is easiest to simply set it to 1.

In conclusion, for calibration we need the budget shares, initial prices, total expenditure, income elasticities and the own-price uncompensated elasticities. From this, we can derive base year consumption volumes, the Allen partial substitution elasticities through equation (B.28), σ (and therefore b) through equation (B.31) and the inversion of the A-matrix, e through equation (B.33) and inversion of the B-matrix, and finally α through equation (B.39).

It is possible that the initial shares and elasticities lead to inconsistent calibrated values for the b or e parameters. One solution, modified from Hertel (1997), is to implement some sort of maximum entropy method—explicitly imposing the constraints on the parameters. Step 1 is to calibrate the b-parameters using the following minimization problem:

$$\min L = \sum_i s_i(\varepsilon_{ii} - \varepsilon^0_{ii})^2$$

subject to

$$\varepsilon_{ii} = (1-b_i)\,(s_i - 1) - s_i\left[b_i + \eta_i - \sum_j s_j b_j\right]$$

$$0 < b_i < 1$$

The loss function is a weighted some of square errors where ε^0 represents the initial or target own-price elasticity and ε will be the estimated elasticity with the constraints holding. The first constraint is a transformation of equation (B.8) where the income elasticity is substituted into the definition of the own-price elasticity (swapping out for the yet unknown e-coefficients). One critical issue is to ascertain what income elasticities to use in the formula above. One could use the target income elasticities, or an initial transformation of the target elasticities such as described above.

The next step calibrates the e-parameters with some target income elasticities as given as well as the now calibrated b-parameters. The minimization problem is formulated as the following:

$$\min L = \sum_i s_i (\eta_i - \eta_i^0)^2$$

subject to

$$\eta_i = \frac{1}{\sum_k s_k e_k} \left[e_i b_i - \sum_k s_k e_k b_k \right] - (b_i - 1) + \sum_k b_k s_k$$

$$\sum_i s_i \eta_i \equiv 1$$

$$(\eta_i - 1)\ \left(\eta_i^0 - 1\right) > 0$$

The final constraint insures that the estimated income elasticities preserve their relationship relative to 1, i.e. target elasticities lower than 1 remain lower than 1 in the estimation procedure.

B.1.4 CDE in first differences

It is useful to decompose changes in demand using a linearized version of the demand function, and that which is used in the standard GEMPACK version of the CDE function. The CDE implicit utility function can be used to derive a relation between changes in income, utility and prices (all in per capita terms). The first step in the differentiation of the utility function, equation (B.1), leads to the following expression:

$$\begin{aligned} 0 &= \sum_i \alpha_i e_i b_i u^{e_i b_i - 1} \left(\frac{p_i}{Y}\right)^{b_i} du \\ &- \sum_i \alpha_i b_i u^{e_i b_i} \left(\frac{p_i}{Y}\right)^{b_i - 1} \frac{p_i}{Y^2} dY \\ &+ \sum_i \alpha_i b_i u^{e_i b_i} \left(\frac{p_i}{Y}\right)^{b_i - 1} \frac{1}{Y} dp_i \end{aligned}$$

This can be simplified by inserting the expression for the demand equation, equation (B.3), and replacing demand with the budget shares (s_i):

$$0 = \frac{du}{u} \sum_i e_i s_i - \frac{dY}{Y} \sum_i s_i + \sum_i s_i \frac{dp_i}{p_i}$$

And the final expression can be written as:

$$\dot{Y} = \sum_i e_i s_i \dot{u} + \sum_i s_i \dot{p}_i \tag{B.40}$$

where the dotted variables represent the percent change (and noting that the sum of the budget shares is equal to 1).

The differentiation of the demand function, equation (B.3) is somewhat more tedious. The first step leads to the following expression:

$$\begin{aligned} dx_i &= \alpha_i b_i e_i b_i u^{e_i b_i - 1} \left(\frac{p_i}{Y}\right)^{b_i - 1} \frac{du}{D} \\ &+ \alpha_i b_i u^{e_i b_i} (b_i - 1) \left(\frac{p_i}{Y}\right)^{b_i - 2} \frac{1}{Y} \frac{dp_i}{D} \\ &- \alpha_i b_i u^{e_i b_i} (b_i - 1) \left(\frac{p_i}{Y}\right)^{b_i - 2} \frac{p_i}{Y^2} \frac{dY}{D} \\ &- \alpha_i b_i u^{e_i b_i} \left(\frac{p_i}{Y}\right)^{b_i - 1} D^{-2} \sum_j \alpha_j b_j e_j b_j u^{e_j b_j - 1} \left(\frac{p_j}{Y}\right)^{b_j} du \\ &- \alpha_i b_i u^{e_i b_i} \left(\frac{p_i}{Y}\right)^{b_i - 1} D^{-2} \sum_j \alpha_j b_j b_j u^{e_j b_j} \left(\frac{p_j}{Y}\right)^{b_j - 1} \frac{1}{Y} dp_j \\ &+ \alpha_i b_i u^{e_i b_i} \left(\frac{p_i}{Y}\right)^{b_i - 1} D^{-2} \sum_j \alpha_j b_j b_j u^{e_j b_j} \left(\frac{p_j}{Y}\right)^{b_j - 1} \frac{p_j}{Y^2} dY \end{aligned}$$

where D is the denominator in the demand equation. This can be simplified to the following expression in terms of the percent changes:

$$
\begin{aligned}
\dot{x_i} &= e_i b_i \dot{u} + (b_i - 1)\,\dot{p_i} - (b_i - 1)\,\dot{Y} \\
&- \sum_j e_j b_j s_j \dot{u} - \sum_j b_j s_j \dot{p_j} + \sum_j b_j s_j \dot{Y}
\end{aligned}
$$

Re-grouping terms, the expression becomes:

$$
\begin{aligned}
\dot{x_i} &= (b_i - 1)\dot{p_i} - \sum_j b_j s_j \dot{p_j} \\
&+ \dot{u}\left[e_i b_i - \sum_j e_j b_j s_j\right] \\
&+ \dot{Y}\left[\sum_j b_j s_j - (b_i - 1)\right]
\end{aligned}
$$

The percent change in u can be replaced with the expression above, equation (B.40), to yield the following after re-arrangement:

$$
\begin{aligned}
\dot{x_i} &= (b_i - 1)\,\dot{p_i} - \sum_j b_j s_j \dot{p_j} - \frac{1}{\sum_k e_k s_k} \sum_j s_j \dot{p_j} \left[e_i b_i - \sum_k e_k b_k s_k\right] \\
&+ \dot{Y}\left[\sum_k b_k s_k - (b_i - 1) + \frac{1}{\sum_k e_k s_k}\left(e_i b_i - \sum_k e_k b_k s_k\right)\right]
\end{aligned}
$$

The final formula inserts the formulas for the income and price elasticities from above to simplify further to the following expression:

$$
\dot{x_i} = \sum_j \varepsilon_{ij} \dot{p_j} + \eta_i \dot{Y} \tag{B.41}
$$

Appendix C

Dynamic model equations with multi-step time periods

The step size in the model scenarios is allowed to vary across time—in order to save compute time and storage. Particularly in the long-run scenarios, annual increments are not particularly useful. Some of the equations in the model—essentially almost any equation that relies on a lagged variable needs to take into account the variable step size, for example the capital stock accumulation equation.

$$KStock = (1-\delta) KStock_{-1} + XF_{Inv,-1}$$

In fact, this equation is not even necessary in the model for a step size of 1 since both variables on the right-hand side of the equation are lags. However, let n be the step-size, eventually 1. Then through recursion, the capital accumulation function becomes:

$$KStock_t = (1-\delta)^n KStock_{t-n} + \sum_{j=1}^{n} (1-\delta)^{j-1} XF_{Inv,t-j}$$

If the model is run in step sizes greater than 1, the intermediate values of real investment are not calculated. They can be replaced by assuming a linear growth model for investment:

$$XF_{Inv,t} = \left(1+\gamma^I\right) XF_{Inv,t-1}$$

Replacing this in the accumulation function yields:

$$KStock_t = (1-\delta)^n KStock_{t-n} + \sum_{j=1}^{n} (1-\delta)^{j-1} \left(1+\gamma^I\right)^{n-j} XF_{Inv,t-n}$$

With some algebraic manipulation, this formula can be reduced to the following:

$$KStock_t = (1-\delta)^n KStock_{t-n} + \frac{\left(1+\gamma^I\right)^n - (1-\delta)^{\,n}}{\gamma^I + \delta} XF_{Inv,t-n}$$

Where we have the following equation to determine the growth rate of investment:

$$XF_{Inv,t} = \left(1+\gamma^I\right)^n XF_{Inv,t-n}$$

which itself is now a function of contemporaneous investment. If n is equal to 1, it is clear that this equation simplifies to the simple 1 step accumulation function. The capital accumulation function is no longer exogenous since it depends on the investment growth rate, which itself is endogenous. To avoid scale problems, equations (3.155) and (3.156), are used in place of the two more transparent equations above. Equation (3.155) is likely to evaluate to somewhere between 10 and 20 since the first term is 1 plus the average annual growth of investment, to which is added the depreciation rate less 1. If investment growth is 5 percent and depreciation is also 5 percent, then the value is 10. The first term on the right-hand side of equation (3.156) is likely to be relatively small since it takes the previous capital stock and subtracts a multiple of the previous period's investment (lagged n years), and then multiplies by the depreciation factor, so that the largest term is the second term, which is a multiple of the current volume of investment.

Appendix D

The accounting framework

This annex provides a visual representation of the accounting framework [to be completed].

Table D.1: The accounting framework for the MANAGE model

	ACT	COMM	LAB
Activities		$\chi^{P}_{a,j} P_{a,j} X_{a,j}$	
Commodities	$\chi^{PA}_{i} PA_i XA_{i,a}$	$\chi^{PA}_{i} PA_i XA_{i,j}$	
Labor	$W_{a,l} L^d_{a,l}$		
Capital	$PK_{a,v} K^d_{a,v} + RKF_a KF_a$		
Value added tax		$\left(\chi^{PD}_{j} PD_j + PMARG_j \zeta^d_j\right)\tau^{va,d}_j XD_j$ $+\left(PM_j + PMARG_j \zeta^m_j\right)\tau^{va,m}_j XM_j$	
Sales tax	$\sum_i \left(\tau^A_{i,a} + \varsigma^A_{i,a}\right)\chi^{PA}_{i} PA_i XA_{i,a}$		
Production tax	$\left(\tau^p_a + \tau^s_a\right) PX_a XP_a$		
Trade tax		$\tau^m_j PWM_j ER(XM_j + STB_j{}^m)$ $+\tau^e_j chi^{PE}_j PE_j XE_j$	
Direct tax			
Enterprises			
Households			$\sum_a W_{a,l} L^d_{a,l}$
Government			
Investment			
Stock change			
Rest of the world		$PWM_j ER(XM_j + STB_j{}^m)$	

Table D.1: The accounting framework for the MANAGE model (cont.)

	CAP	VAT	ATX	PTX	MTX	DTX
Activities						
Commodities						
Labor						
Capital						
Value added tax						
Sales tax						
Production tax						
Trade tax						
Direct tax	$\sum_a \tau_a^k RKF_a KF_a$					
Enterprises	$\chi_{entr}^k KAPY$					
Households	$\chi_h^k KAPY$					
Government	$KAPY\left(1-\sum\limits_{entr}\chi_{entr}^k-\sum\limits_h \chi_h^k\right)$	YG_{otx}	YG_{atx}	YG_{ptx}	YG_{mtx}	YG_{dtx}
Investment						
Stock change						
Rest of the world						

Table D.1: The accounting framework for the MANAGE model (cont.)

	ENTR	HH	GOV
Activities			
Commodities		$\chi_i^{PA} PA_{i,h} XA_{i,h}$	$\chi_i^{PA} PA_{i,Gov} XA_{i,Gov}$
Labor			
Capital			
Value added tax			
Sales tax		$\sum_i \left(\tau_{i,h}^A+\varsigma_{i,h}^A\right)\chi_i^{PA} PA_i XA_{i,h}$	$\sum_i \left(\tau_{i,Gov}^A+\varsigma_{i,Gov}^A\right)\chi_i^{PA} PA_i XA_{i,Gov}$
Production tax			
Trade tax			
Direct tax	Tax_{entr}^{entr}	Tax_h^h	Tax^{Gov}
Enterprises	$Transfers_{entr,entr}$	$Transfers_{h,entr}$	$Transfers_{Gov,entr}$
Households	$Transfers_{h,entr}$	$Transfers_{h,h}$	$Transfers_{Gov,h}$
Government	$Transfers_{entr,Gov}$	$Transfers_{h,Gov}$	$Transfers_{Gov,Gov}$
Investment	S_{entr}^{entr}	S_h^h	S^g
Stock change			
Rest of the world	$Transfers_{entr,RoW}$	$Transfers_{h,RoW}$	$Transfers_{Gov,RoW}$

Table D.1: The accounting framework for the MANAGE model (cont.)

	INV	**STB**	**RoW**
Activities			
Commodities	$\chi_i^{PA} PA_{i,Inv} XA_{i,Inv}$	$\chi_i^{PS} PS_i STB_i^d + PM_i STB_i^m$	$ER \chi_i^{PWE} PWE_{i,Inv} XE_{i,Inv}$
Labor			
Capital			
Value added tax			
Sales tax	$\sum_i \left(\tau_{i,Inv}^A + \varsigma_{i,Inv}^A\right) \chi_i^{PA} PA_i XA_{i,Inv}$		
Production tax			
Trade tax			
Direct tax			
Enterprises			$Transfers_{RoW,entr}$
Households			$Transfers_{RoW,h}$
Government			$Transfers_{RoW,Gov}$
Investment			ERS^f
Stock change	$\sum_i \chi_i^{PS} PS_i STB_i^d + PM_i STB_i^m$		
Rest of the world			

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